BRAUER'S HEIGHT ZERO CONJECTURE FOR TWO PRIMES HOLDS TRUE

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ABSTRACT. In this paper we complete the proof of Brauer's height zero conjecture for two primes proposed by G. Malle and G. Navarro.

1. Introduction

The famous Brauer's height zero conjecture [4] is one of the most important open problems in the modular representation theory of finite groups. Up to now, the "if" part of Brauer's height zero conjecture was proved by Kessar-Malle [12], and the "only if" part was reduced to checking the so-called inductive Alperin-McKay condition for all simple groups by Navarro-Späth [23]. In addition, Brauer's height zero conjecture is known to hold true for quasi-simple groups by Kessar-Malle [13].

Among the classes of blocks, the trueness of Brauer's height zero conjecture for the 2-blocks of maximal defect and for blocks with meta-cyclic defect groups was shown by Navarro-Tiep [24] and Sambale [27], respectively. Also, Brauer's height zero conjecture for principal blocks was proved by Malle-Navarro [18]. Very recently, the conjecture has been finally proved by Malle-Navarro-Schaeffer Fry-Tiep [19].

Motivated by Brauer's height zero conjecture, G. Malle and G. Navarro put forward the following conjecture:

Conjecture 1.1. [17, Conjecture A] *Let G be a finite group, and let p and q be primes. Then the elements of some Sylow p-subgroup of G commute with the elements of some Sylow q-subgroup of G if and only if the characters in $B_p(G)$ have degree not divisible by q and the characters in $B_q(G)$ have degree not divisible by p, where $B_p(G)$ denotes the set of irreducible complex characters in the principal p-block of G.*

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In [17], the authors proved the "only if" part of Conjecture 1.1 in full generality, and the "if" part of Conjecture 1.1 under the assumption that the inductive Alperin-McKay condition holds true for principal blocks of non-abelian simple groups. In this paper, we remove their assumption and provide a direct proof of the "if" part.

Theorem 1.2. *Conjecture 1.1 holds true.*

The proof of Theorem 1.2 depends on the classification theorem of finite simple groups. Also, the following result about almost simple groups with p-automorphisms turns out to be crucial.

Theorem 1.3. *Let p, q be different primes and let $S \leq A \leq \mathrm{Aut}(S)$, where S is a nonabelian simple group with $q \mid |S|$. If $|A/S| = p^a$ with a positive integer a and S has a nilpotent Hall $\{p, q\}$-subgroup, then exactly one of the following holds:*

(1) *A has a nilpotent Hall $\{p, q\}$-subgroup; or*
(2) *the conjugation action of P on $B_q(S)$ has a nontrivial orbit, where $P \in \mathrm{Syl}_p(A)$.*

2. Proof of Theorem 1.2

Lemma 2.1. *Let G be a finite group, and let p and q be primes. Suppose that $G = NP$, where $N \trianglelefteq G$ and P is a p-subgroup of G. If N has a p-nilpotent Hall $\{p, q\}$-subgroup, then P normalizes a Sylow q-subgroup of N.*

Proof. We first suppose that $q = p$. Let R be a Sylow q-subgroup of G containing P. Clearly, $R \cap N$ is a Sylow q-subgroup of N, $R \cap N \trianglelefteq R$ and so P normalizes $R \cap N$.

We now assume that $q \neq p$. Let QP_0 be a p-nilpotent Hall $\{p, q\}$-subgroup of N, where $Q \in \mathrm{Syl}_q(N)$ and $P_0 \in \mathrm{Syl}_p(N)$. Then $P_0 \in N_G(Q)$, and by Frattini's argument, we have $G = NN_G(Q)$. Let P_1 be a Sylow p-subgroup of $N_G(Q)$ containing P_0. Then P_1 is also a Sylow p-subgroup of G since $G = NP$. Hence there is some $g \in G$ such that $P \leq P_1^g$, and therefore P normalizes Q^g, as desired. □

Remark 2.2. The conclusion of Lemma 2.1 does not hold true under the assumption that N has a q-nilpotent Hall $\{p, q\}$-subgroup. As an example we may take $G = S_4, N = A_4, p = 2, q = 3$ and $P = \langle(1234)\rangle$.

For our purpose, we need the so-called generalized p'-core $\mathrm{O}_{p^*}(G)$ of a finite group G, which is defined by $\mathrm{O}_{p^*}(G) = \langle N \mid N \trianglelefteq G, N \text{ is a } p^*\text{-group}\rangle$. Here a finite group G is called a p^*-group if the following two conditions hold.

(i) $\mathrm{O}^p(G) = G$, i.e., G does not have a nontrivial p-factor group.
(ii) Whenever $N \trianglelefteq G$ and $P \in \mathrm{Syl}_p(N)$, then $G = NC_G(P)$.

It is known that $\mathrm{O}_{p'}(G) \leq \mathrm{O}_{p^*}(G)$ and for the layer $E(G)$ of G we have $E(G) \leq \mathrm{O}_{p^*}(G)$. Moreover, $\mathrm{O}_{p^*}(G)$ is the largest normal p^*-subgroup of G and G is p-constrained if and only if $\mathrm{O}_{p^*}(G) = \mathrm{O}_{p'}(G)$. For more details, see [3] or [10, Chap. X, §14]. Sometimes, $E(G)$ is written as $\mathrm{O}_E(G)$ for the purpose of convenience.

Let $f_{0,p}(G) = \sum_{g \in G} f_g g$ be the block idempotent of the principal p-block of G over a splitting field of characteristic p, and let $\mathrm{O}_{f_{0,p}}(G)$ be the subgroup of G generated by $\mathrm{supp}(f_{0,p}(G)) = \{g \mid f_g \neq 0\}$. An important fact for us is that $\mathrm{supp}(f_{0,p}(G)) \leq \mathrm{O}_{p^*}(G)$ (see [31]).

In addition, we will freely use a theorem of Wielandt [30], stating that if G has a nilpotent Hall $\{p,q\}$-subgroup, then all Hall $\{p,q\}$-subgroups of G are conjugate and each $\{p,q\}$-subgroup of G is contained in some Hall $\{p,q\}$-subgroup of G.

Theorem 2.3. *Let G be a finite group, and let p and q be primes. Then the elements of some Sylow p-subgroup of G commute with the elements of some Sylow q-subgroup of G if and only if the characters in $B_p(G)$ have degree not divisible by q and the characters in $B_q(G)$ have degree not divisible by p.*

Proof. By a recent result of Malle and Navarro [18, Theorem A], we may assume that $p \neq q$ and $pq \mid |G|$. Since the "only if" part of the theorem has been proved in [17, Theorem 4.1], it suffices to prove the "if" part of the theorem.

We use induction on the order $|G|$ of G. Notice that the hypotheses are inherited by factor groups and normal subgroups, and that the assertion is equivalent to prove that G has a nilpotent Hall $\{p,q\}$-subgroup. Clearly, $\mathrm{O}_{\{p,q\}'}(G) = 1$ by induction.

(1) *We may assume that G has a unique minimal normal subgroup M.*

Proof. Let N be a minimal normal subgroup of G. If $p \mid |G/N|$ but $q \nmid |G/N|$, then any Sylow p-subgroup of G/N is a nilpotent Hall $\{p,q\}$-subgroup of G/N, while if $pq \mid |G/N|$, then G/N has a nilpotent Hall $\{p,q\}$-subgroup by induction.

Therefore, if $N \neq M$ are two minimal normal subgroups of G, then both G/N and G/M have nilpotent Hall $\{p,q\}$-subgroups. According to [26, Corollary 8], $G = G/(N \cap M)$ has a Hall $\{p,q\}$-subgroup, say H. Now, by Wielandt, H is contained in a Hall $\{p,q\}$-subgroup of $G/N \times G/M$, which is nilpotent. Thus H is a nilpotent Hall $\{p,q\}$-subgroup of G. □

(2) *We may assume that $G = \mathrm{O}_{q^*}(G)P\,\mathrm{O}_{p^*}(G)Q$ with $\mathrm{O}_{p^*}(G)Q \trianglelefteq G$ and $\mathrm{O}_{q^*}(G)P \trianglelefteq G$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$.*

Proof. It immediately follows from [31] that $\mathrm{Irr}(G/\mathrm{O}_{p^*}(G)) \subseteq B_p(G)$. Therefore all irreducible characters of $G/\mathrm{O}_{p^*}(G)$ have degrees not divisible by q, and so $G/\mathrm{O}_{p^*}(G)$ has an abelian normal Sylow q-subgroup by [21, Theorem 2.3]. If $Q \in \mathrm{Syl}_q(G)$, then $\mathrm{O}_{p^*}(G)Q \trianglelefteq G$. Similarly, we have $\mathrm{O}_{q^*}(G)P \trianglelefteq G$, where $P \in \mathrm{Syl}_p(G)$. If $W :=$

$\mathrm{O}_{q^*}(G)P\,\mathrm{O}_{p^*}(G)Q \lhd G$, then W and so G has a nilpotent Hall $\{p,q\}$-subgroup by induction. So we may assume that $G = \mathrm{O}_{q^*}(G)P\,\mathrm{O}_{p^*}(G)Q$, as claimed. □

(3) *We may assume that* $K := \mathrm{O}_{p^*}(G) \cap \mathrm{O}_{q^*}(G) \neq 1$. *In particular, we have* $M \leq K$.

Proof. If $\mathrm{O}_{p^*}(G) \cap \mathrm{O}_{q^*}(G) = 1$, then either $\mathrm{O}_{p^*}(G) = 1$ or $\mathrm{O}_{q^*}(G) = 1$, by the uniqueness of the minimal normal subgroup M of G. Without loss of generality, we may assume that $\mathrm{O}_{p^*}(G) = 1$, hence $Q \lhd G$. Thus we have $Q \leq \mathrm{O}_{p'}\,G) \leq \mathrm{O}_{p^*}(G) = 1$, a contradiction since $pq \mid |G|$. □

(4) *We may assume that either* $\mathrm{O}_{q^*}(G)P = G$ *or* $\mathrm{O}_{p^*}(G)Q = G$.

Proof. Suppose that $\mathrm{O}_{q^*}(G)P \lhd G$ and $\mathrm{O}_{p^*}(G)Q \lhd G$. By induction, we may assume that $PM/M \times QM/M$ is a nilpotent Hall $\{p,q\}$-subgroup of G/M with the replacement of a suitable conjugate of Q if necessary. In particular, $[P,Q] \subseteq M$. Also, since $\mathrm{O}_{p^*}(G)Q \lhd G$, it follows that $\mathrm{O}_{p^*}(G)Q$ has a nilpotent Hall $\{p,q\}$-subgroup by induction. This implies that MQ has a nilpotent Hall $\{p,q\}$-subgroup. Note that P normalizes MQ. Now, replacing N by MQ and G by $(MQ)P$ in Lemma 2.1, we conclude that P normalizes Q^x for some $x \in M$. So, $[P,Q^x] \subseteq Q^x$. Since $Q^xM = QM$ and $PM/M \times QM/M$ is nilpotent, it follows that $[P,Q^x] \subseteq M$. Hence

$$[P,Q^x] \subseteq Q^x \cap M. \tag{2.1}$$

On the other hand, $PQ^x \cap \mathrm{O}_{q^*}(G)P$ is a Hall $\{p,q\}$-subgroup of $\mathrm{O}_{q^*}(G)P$ which is nilpotent by induction, since $\mathrm{O}_{q^*}(G)P$ is a proper normal subgroup of G. Therefore, $P(Q^x \cap \mathrm{O}_{q^*}(G))$ is nilpotent, and so

$$[P,Q^x \cap M] = 1. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain $[P,Q^x] = 1$ by [2, Appendix A, Lemma A.2]. Thus G has a nilpotent Hall $\{p,q\}$-subgroup, and we are done. □

From now on we suppose that $G = \mathrm{O}_{q^*}(G)P$.

(5) *We may assume that* $F(G) = 1$.

Proof. If M is a q-group, then $M \leq Z(\mathrm{O}_{q^*}(G))$, by [10, Chap. X, Lemma 14.3 c)]. Hence for any $\lambda \in \mathrm{Irr}(M)$, there is some $\theta \in B_q(\mathrm{O}_{q^*}(G))$ such that $\theta = \theta(1)\lambda$ by [22, Theorem 9.4] and Clifford's Theorem [11, Theorem 6.2]. If the inertia subgroup of θ in G does not contain a Sylow p-subgroup of G, then all irreducible constituents of θ^G have degree divisible by p by the Clifford correspondence [11, Theorem 6.11]. In particular, $B_q(G)$ has an irreducible character of degree divisible by p, a contradiction. So θ and hence λ is G-invariant. In particular, λ is P-invariant. It follows, by Brauer's permutation lemma, that

$$[P, M] = 1. \tag{2.3}$$

By induction on G/M, we have

$$[P, Q] \subseteq M \tag{2.4}$$

after replacing a conjugate of Q if necessary. Thus $[P, Q] = 1$ by [2, Appendix A, Lemma A.2], and so G has a nilpotent Hall $\{p, q\}$-subgroup.

Now, suppose that M is a p-group. Again by induction on G/M, we have

$$[Q, P] \subseteq M, \tag{2.5}$$

after replacing a conjugate of P if necessary.

If $\mathrm{O}_{p^*}(G)Q \lhd G$, then by induction, $\mathrm{O}_{p^*}(G)Q$ has a nilpotent Hall $\{p, q\}$-subgroup. Hence

$$[Q, M] = 1. \tag{2.6}$$

Thus $[Q, P] = 1$ by [2, Appendix A, Lemma A.2], and so G has a nilpotent Hall $\{p, q\}$-subgroup.

Finally, suppose that $G = \mathrm{O}_{p^*}(G)Q$. Repeating the above argument as for the case where M is a q-group with replacement of q by p, we obtain that G has a nilpotent Hall $\{p, q\}$-subgroup. Thus we may assume that $F(G) = 1$, as desired. □

By the uniqueness of the minimal normal subgroup M of G, the layer $\mathrm{O}_E(G)$, which is indeed equal to M, is a direct product of some copies of a nonabelian simple group, say S, since $F(G) = 1$. Note that either $p \mid |S|$ or $q \mid |S|$ since $\mathrm{O}_{\{p,q\}'}(G) = 1$.

(6) *The theorem holds if $q \mid |S|$.*

Proof. Suppose that $q \mid |S|$. Then $\mathrm{O}_{q'}(G) = 1$, since otherwise $M \leq \mathrm{O}_{q'}(G)$, and so $q \nmid |M|$, a contradiction. Write $L = \mathrm{O}_{q^*}(G)$. According to [10, Chap. X, Theorem 14.17], we have

$$G = LP = \mathrm{O}_{q',E}(L)\, \mathrm{O}_{q^*}(C_L(Q_0))P = \mathrm{O}_E(L)\, \mathrm{O}_{q^*}(C_L(Q_0))P$$

where $Q_0 \in \mathrm{Syl}_q(\mathrm{O}_{q',E,q}(L))$ and $\mathrm{O}_{q',E}$ and $\mathrm{O}_{q',E,q}$ are as in [10, Chap. X, Definition 14.16]. Since $C_L(Q_0)$ normalizes each direct factor of $\mathrm{O}_E(L) = \mathrm{O}_E(G)$, we deduce that all direct factors of $\mathrm{O}_E(L)$ are normal in L. Assume that some factor S_i of $\mathrm{O}_E(L)$ is not normalized by any Sylow p-subgroup of G. Let θ be a nontrivial irreducible character of the principal q-block of S_i. Considering the inertial subgroup of the irreducible character $\phi := 1 \times \cdots \times \theta \times \cdots \times 1$ of the principal q-block of $\mathrm{O}_E(L)$, we conclude that all irreducible constituents of ϕ^G have degree divisible by p. In particular, $B_q(G)$ has an irreducible character of degree divisible by p, a contradiction. Therefore S_i is normalized by P^x for some $x \in G$, and so S_i is normal in G, since $G = LP^x$. Thus $\mathrm{O}_E(L) = S_i$ and G is an almost simple group with socle $S_i \cong S$.

By [17, Theorem 5.1], we may assume that $G \neq S$. According to [10, Theorem 14.18] and the first sentence of its following Remarks 14.19 we obtain $L = S\,\mathrm{O}_{q'}(C_L(Q_0))$. Note that S has a nilpotent Hall $\{p, q\}$-subgroup by induction. If $G = L$, then G has a nilpotent Hall $\{p, q\}$-subgroup and we are done.

In the following we let $L < G$. Applying Theorem 1.3, we may assume that the conjugation action of P on $B_q(S)$ has a nontrivial orbit. By [1, Lemma 1], the restriction of characters of L to S induces a bijection between $B_q(L)$ and $B_q(S)$. Hence the conjugate action of P on $B_q(L)$ also has a nontrivial orbit. This implies that $B_q(G)$ has an irreducible character of degree divisible by p, a contradiction. □

(7) *The theorem holds if* $q \nmid |S|$.

Proof. If $q \nmid |S|$, then $p \mid |S|$ and $q \nmid |M|$. First, suppose that $\mathrm{O}_{p^*}(G)Q \neq G$. Replacing N by $\mathrm{O}_{p^*}(G)Q$ in Lemma 2.1, we may assume that P normalizes Q^x for some $x \in G$, i.e., $[P, Q^x] \subseteq Q^x$. However, PQ^xM/M is a Hall $\{p, q\}$-subgroup of G/M which is nilpotent by induction. Hence $[P, Q^x] \subseteq M$, and therefore $[P, Q^x] \subseteq Q^x \cap M = 1$. Thus G has a nilpotent Hall $\{p, q\}$-subgroup.

Finally suppose that $G = \mathrm{O}_{p^*}(G)Q$. Repeating the argument of (6) with the role of q replaced by p, we deduce that the theorem holds. □

□

3. Almost simple groups

For simple groups S of Lie type other than the Tits simple group, we introduce the following setup. Let $\mathbf{G}$ be a simple algebraic group of adjoint type over an algebraically closed field of characteristic r and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism, with finite group of fixed points $G := \mathbf{G}^F$ such that $S = G'$.

If F is a Frobenius endomorphism, then it defines an $\mathbb{F}_{r_1}$-rational structure on $\mathbf{G}$ for some power r_1 of the characteristic r. In the case that F is not a Frobenius endomorphism we let r_1 be the absolute value of all eigenvalues of F^2 on the character group of an F-stable maximal torus of $\mathbf{G}$; it is an integral power of the characteristic as well (see [20, §22.1] or [28, §11.6 and Remark 11.15]).

According to [9, Theorem 2.5.1], $\mathrm{Aut}(S)$ is generated by the inner automorphisms, diagonal automorphisms, field automorphisms and graph automorphisms of S. Furthermore, by [9, Lemma 2.5.8.(a)], the group G is exactly the subgroup of $\mathrm{Aut}(S)$ generated by S and its diagonal automorphisms.

To proceed, we introduce some notation. For an integer n we write n_p for the largest power of p dividing n, and for a group H and $x \in H$, we denote by x^H the conjugacy class of x in H. If q is a prime, which is coprime to r_1, we denote by $e := e_q(r_1)$ the multiplicative order of r_1 modulo q if $q \neq 2$, respectively

$$e_2(r_1) = \begin{cases} 1 & \text{if } r_1 \equiv 1 \pmod 4, \\ 2 & \text{if } r_1 \equiv -1 \pmod 4. \end{cases}$$

In addition, for a positive integer m let $\Phi_m(x) \in \mathbb{Z}[x]$ denote the cyclotomic polynomial whose roots are the primitive m-th roots of unity.

Proposition 3.1. *Let S be a finite simple group of Lie type other than the Tits simple group, and let $(\mathbf{G}, F)$ be as above such that $S = G'$. Let p and q be different primes with $q \neq r$ and $q \mid |S|$. Suppose that S has a nilpotent Hall $\{p, q\}$-subgroup, and that $S \leq A \leq \mathrm{Aut}(S)$ with $|A/S| = p^a$ for some positive integer a. If A does not have a nilpotent Hall $\{p, q\}$-subgroup, then $A = S\langle\phi\rangle$ for some field automorphism ϕ of S (and also of G), and S has a q-element s such that the conjugacy class s^G of s in G is not fixed by the conjugation action of A on S.*

Proof. Let $e = \mathrm{ord}_q(r_1)$. We first prove the first conclusion of the proposition. It is true if $p \nmid |S|$ by [9, Theorem 7.1.2].

So we may assume that $p \mid |S|$. Since S has a nilpotent Hall $\{p, q\}$-subgroup, it follows by [17, Proposition 3.4] that p and q are not the defining characteristic of S. Using the duality between $\mathbf{G}$ and its simply-connected correspondent, we get by [17, Lemma 3.1 and Proposition 3.5] that p, q are odd and a Hall $\{p, q\}$-subgroup of S is abelian and contained in some Sylow Φ_e-torus of $\mathbf{G}$ with $e = \mathrm{ord}_q(r_1) = \mathrm{ord}_p(r_1)$.

Note that only $S = D_4(r_1)$ has a graph automorphism of odd order, namely 3. But for $p = 3$, the Sylow 3-subgroups of $D_4(r_1)$ are not abelian, a contradiction. Suppose that A induces diagonal automorphisms on S. Then

- (i) $S = A_n(r_1)$ and $p \mid (n+1, r_1 - 1)$,
- (ii) $S = {}^2A_n(r_1)$ and $p \mid (n+1, r_1 + 1)$,
- (iii) $S = E_6(r_1)$ and $p = 3 \mid (r_1 - 1)$, or
- (iv) $S = {}^2E_6(r_1)$ and $p = 3 \mid (r_1 + 1)$.

In the cases (i) and (iii) we get $e = 1$, and in the cases (ii) and (iv) $e = 2$. By [15, Proposition 2.2], $\Phi_e(r_i)$ is the unique cyclotomic polynomial factor in $|G|$ divisible by p. According to [16, Lemma 5] we have $p \mid \Phi_{ep}(r_1)$. Since in the cases (i) and (ii) the polynomial $\Phi_{ep}(r_1)$ also occurs in $|G|$ we obtain a contradiction by the uniqueness of $\Phi_e(r_i)$ which is divisible by p. Also, since in the cases (iii) and (iv), $\Phi_3(r_1)$ and $\Phi_6(r_1)$ occur in $|G|$ we obtain a contradiction by the same argument. Thus $A = S\langle\phi\rangle$ for some field automorphism ϕ of S, as desired.

Before we proceed with the proof of the second conclusion we collect some useful facts. First notice that in all cases the prime p is odd. Furthermore,

$$|G|_{r'} = \prod_{m\in\Delta} \Phi_m(r_1)^{a_m},$$

where Δ is a set of some positive integers and $a_m > 0$ for all $m \in \Delta$ (see [8, Section 10-1]). By [8, Section 9-1],

$$|C_G(\phi)|_{r'} = \prod_{m\in\Delta} \Phi_m(r_0)^{a_m},$$

where $r_1 = r_0^{p^a}$.

Clearly, ϕ does not centralize any Sylow q-subgroup of S since A does not have a nilpotent Hall $\{p, q\}$-subgroup. Hence

$$|C_G(\phi)|_q < |G|_q.$$

Let $e_0 = e_q(r_0)$. Since $r_1 = r_0^{p^a}$, it follows by the formula for orders of elements and their powers in a group that $e_0 = e \cdot \gcd(p^a, e_0)$. Assume that $q \mid \Phi_e(r_0)$. Then $e_0 \le e$ and so $e_0 = e$. Since by [14, Lemma 5.2 (a)] $\Phi_e(x)$ is the only cyclotomic factor of $x^{ep^a} - 1$ with $q \mid \Phi_e(r_0)$, we see that

$$\Phi_e(r_0)_q = \Phi_e(r_1)_q.$$

Furthermore, by [16, Lemma 5], $q \mid \Phi_m(r_0)$ if and only if $m = eq^j$ for some $j \ge 0$, if and only if $q \mid \Phi_m(r_1)$, in which case $(\Phi_m(r_0)^{a_m})_q = (\Phi_m(r_1)^{a_m})_q$. Thus

$$|C_G(\phi)|_q = |G|_q,$$

a contradiction. Therefore $e_0 > e$ (i.e., $\gcd(p^a, e_0) \ne 1$) and so e_0 is divisible by pe. In particular, $p \mid e_0$ and $e_0 \ge p$. Also, q is odd since otherwise r_0 is odd and so $e_0 = 1$ or 2, a contradiction.

In order to prove now the second conclusion of the proposition, we first deal with classical simple groups, so that S is of type $A_n(r_1)$, ${}^2A_n(r_1)$, $B_n(r_1)$, $C_n(r_1)$, $D_n(r_1)$ or ${}^2D_n(r_1)$ and ϕ is induced by a power of the Frobenius automorphism of $\overline{\mathbb{F}_r}$ sending each element to its rth-power. Let ξ be a primitive $(r_1^e - 1)_q$-th root of unity.

In the following the existence of the chosen x follows immediately from the structure of maximal tori which have been determined in [5].

Type $A_n(r_1)$**.** In this case we have $S = \mathrm{PSL}_{n+1}(r_1)$, $G = \mathrm{PGL}_{n+1}(r_1)$, and $\Delta = \{1, \ldots, n+1\}$. Let $^- : \mathrm{GL}_{n+1}(r_1) \to G$ be the natural epimorphism. Note that $e \le \frac{n+1}{3}$, since $3e \le pe \mid e_0 \le n+1$.

First, we consider the case $e = 1$, hence $n \ge 2$. We put $x = \mathrm{diag}(\xi, \xi^{-1}, 1, \ldots, 1) \in \mathrm{SL}_{n+1}(r_1)$. Thus

$$\phi(x) = \mathrm{diag}(\xi^{r_0}, \xi^{-r_0}, 1, \ldots, 1) = x^{r_0}$$

and $\overline{x}, \overline{\phi(x)} \in S$. We claim that $\overline{x}$ and $\overline{\phi(x)}$ are not conjugate in G. If they were conjugate, then there exists $z = \mathrm{diag}(\lambda, \lambda, \ldots, \lambda) \in Z(\mathrm{GL}_{n+1}(r_1))$ such that xz and $\phi(x)$ are conjugate in $\mathrm{GL}_{n+1}(r_1)$. Thus $\det(z) = \lambda^{n+1} = 1$ and xz and $\phi(x)$ have the same eigenvalues.

If $n > 2$, then $\lambda = 1$ by comparing the eigenvalues of x and $\phi(x)$. Thus $\xi = \xi^{\pm r_0}$, and so $e_0 = 1, 2 < p$, a contradiction. In the case $n = 2$ we have $\{\lambda\xi, \lambda\xi^{-1}, \lambda\} = \{\xi^{r_0}, \xi^{-r_0}, 1\}$. Hence $\lambda = 1$ (contradiction as above) or $1 \neq \lambda = \xi^{\pm r_0}$. Thus $1 = \lambda^3 = \xi^{\pm 3r_0}$. Since $q \nmid r_0$, we get $\xi^3 = 1$, hence $q = 3$ and $e_0 = 1, 2$, a contradiction.

Now let $e > 1$. Let $x \in \mathrm{SL}_{n+1}(r_1)$ be similar to

$$\mathrm{diag}(\xi, \xi^{r_1}, \ldots, \xi^{r_1^{e-1}}, 1, \ldots, 1)$$

in $\mathrm{GL}_{n+1}(\overline{\mathbb{F}_r})$. Then $\overline{x} \in S$ and $\phi(x)$ is similar to

$$\mathrm{diag}(\xi^{r_0}, \xi^{r_0 r_1}, \ldots, \xi^{r_0 r_1^{e-1}}, 1, \ldots, 1)$$

in $\mathrm{GL}_{n+1}(\overline{\mathbb{F}_r})$.

Suppose that $\overline{x}$ and $\overline{\phi(x)}$ are conjugate in $G = \mathrm{PGL}_{n+1}(r_1)$. Then xz and $\phi(x)$ are conjugate in $\mathrm{GL}_{n+1}(r_1)$. By comparing their eigenvalues we see that $\lambda = 1$. Since $e \leq \frac{n+1}{3}$ and $\xi^{r_0} = \xi^{r_1^j}$ for some $0 \leq j \leq e-1$, it follows that $\xi^{r_0^{p^a j-1}-1} = 1$, and so $q \mid (r_0^{p^a j-1} - 1)$. This implies $e_0 \mid (p^a j - 1)$, a contradiction to $p \mid e_0$.

Hence $\overline{x}$ and $\overline{\phi(x)}$ are not conjugate in G, that is, ϕ does not fix the conjugacy class $\overline{x}^G$ of $\overline{x}$ in G.

Type ${}^2A_n(r_1)$**.** In this case we have $S = \mathrm{PSU}_{n+1}(r_1)$ and $G = \mathrm{PGU}_{n+1}(r_1)$, where $n \geq 2$. Let $^- : \mathrm{GU}_{n+1}(r_1) \to G$ denote the natural epimorphism.

If $e = 2$ we put $x = \mathrm{diag}(\xi, \xi^{-1}, 1, \ldots, 1)$, if $e = 1$ we choose $x \in \mathrm{SU}_{n+1}(r_1)$ such that x is $\mathrm{GL}_{n+1}(\overline{\mathbb{F}_r})$-conjugate to $\mathrm{diag}(\xi, \xi^{-1}, 1, \ldots, 1)$, and for $e > 2$ we choose $x \in \mathrm{SU}_{n+1}(r_1)$ such that x is $\mathrm{GL}_{n+1}(\overline{\mathbb{F}_r})$-conjugate to $\mathrm{diag}(\xi, \xi^{-r_1}, \ldots, \xi^{(-r_1)^{e-1}}, 1, \ldots, 1)$. Again we suppose that $\overline{x}$ and $\overline{\phi(x)}$ are conjugate in G.

In the cases $e = 1, 2$, we may argue as in the case $A_n(r_1)$ for $e = 1$ to get a contradiction. Finally let $e > 2$. Since xz is not a q-element whenever $\lambda \neq 1$, we see that $\overline{x}$ and $\overline{\phi(x)}$ are G-conjugate if and only if x and $\phi(x)$ are $\mathrm{GL}_{n+1}(\overline{\mathbb{F}_r})$-conjugate (i.e., $\lambda = 1$). Thus it follows that $\xi^{r_0} = \xi^{(-r_1)^j}$, or equivalently,

$$\xi^{r_0(r_0^{p^a j-1} - (-1)^j)} = 1$$

for some $1 \leq j \leq e-1$. Hence $2 \neq p \mid e_0 \mid 2(p^a j - 1)$, a contradiction.

Type $C_n(r_1)$**.** In this case, $S = \mathrm{PSp}_{2n}(r_1)$ and $G = \mathrm{PCSp}_{2n}(r_1)$, where $n > 1$. Observe that q-elements of S are conjugate in G if and only if their pre-images in $\mathrm{Sp}_{2n}(r_1)$ with the same order are conjugate in $\mathrm{CSp}_{2n}(r_1)$.

Let $x \in \mathrm{Sp}_{2n}(r_1)$ be such that x is $\mathrm{GL}_{2n}(\overline{\mathbb{F}_r})$-conjugate to

$$\mathrm{diag}(D, 1, \ldots, 1, D^{-1}, 1, \ldots, 1),$$

where $D = \mathrm{diag}(\xi, \xi^{r_1}, \ldots, \xi^{r_1^{e-1}})$. The choice of D is possible as $e \leq \frac{2n}{3}$.

Now if $\overline{x}$ and $\overline{\phi(x)}$ are conjugate in G, then

$$\begin{aligned} \xi &= \xi^{\pm r_0 r_1^j} \quad \text{for some } 0 \leq j \leq e-1, \\ &= \xi^{\pm r_0^{p^a j+1}}. \end{aligned}$$

Thus $e_0 \mid 2(p^a j + 1)$, a contradiction since $2 \neq p \mid e_0$.

Type $B_n(r_1)$**.** In this case, we have $S = \Omega_{2n+1}(r_1)$ and $G = \mathrm{SO}_{2n+1}(r_1)$, where $n > 1$. Furthermore, we may assume that r_1 is odd since otherwise $\mathrm{SO}_{2n+1}(r_1) \cong \mathrm{Sp}_{2n}(r_1)$ which has been already proved in the previous case.

The argument now runs exactly as in the case $C_n(r_1)$, where $x \in S$ is $\mathrm{GL}_{2n+1}(\overline{\mathbb{F}_r})$-conjugate to

$$\mathrm{diag}(1, D, 1, \ldots, 1, D^{-1}, 1, \ldots, 1)$$

with $D = \mathrm{diag}(\xi, \xi^{r_1}, \ldots, \xi^{r_1^{e-1}})$.

Type $D_n(r_1)$ or ${}^2D_n(r_1)$, where $n \geq 4$. Now we have $S = \mathrm{P}\Omega_{2n}^{\pm}(r_1)$ and $G = \mathrm{P}(\mathrm{CO}_{2n}^{\pm}(r_1)^{\circ})$, and we can argue as in the case $C_n(r_1)$ with the same x. The existence of x can be guaranteed by [5, Section 4 and 5].

We now handle simple groups of exceptional type, so that S is of type ${}^2B_2(r_1)$, ${}^2G_2(r_1)$, ${}^2F_4(r_1)$, ${}^3D_4(r_1)$, $G_2(r_1)$, $F_4(r_1)$, $E_6(r_1)$, ${}^2E_6(r_1)$, $E_7(r_1)$ or $E_8(r_1)$. For convenience, we list the order of G with Δ in Table 1, where $\Phi_i = \Phi_i(r_1)$. Also, we collect the possible lower bound for p in the last column of Table 1 if $p \nmid |S|$. For instance, if $S = G_2(r_1)$ then $2 \cdot 3 \mid |S|$, so if $p \nmid |S|$ then $p \geq 5$. In addition, if $S = {}^2B_2(r_1)$, then $3 \nmid |S|$ and so if $p \nmid |S|$, then $p \geq 3$.

We claim that $|C_G(\phi)|_q = 1$. Assume that $q \mid |C_G(\phi)|$. Then we have $e_0 \in \Delta$.

We first suppose that $p \nmid |S|$. Note that $pe \mid e_0$. If $S = G_2(r_1)$, then the possibilities for e_0 are 3 or 6 by Table 1, both of which contradict the fact that $p \geq 5$. For the remaining cases, a similar argument follows. So the claim holds in this case.

We now assume that $p \mid |S|$. Recall that $\Phi_e(r_1)$ is the unique cyclotomic polynomial factor of the order polynomial of G which is divisible by p (and q). Furthermore, e_0 is divisible by ep. Clearly, $S \neq {}^2B_2(r_1)$ since p is odd.

Suppose that $S = G_2(r_1)$. By Table 1, we see that $e_0 = 3$ or 6 and so $p = 3$. Since $p \neq r$, it follows by [16, Lemma 5] that p divides either $\Phi_1(r_1)$ and $\Phi_3(r_1)$ or $\Phi_2(r_1)$ and $\Phi_6(r_1)$, a contradiction.

Suppose that $S = {}^3D_4(r_1)$. Since $e_0 \in \{3, 6, 12\}$, we again have $p = 3$, which similarly leads to a contradiction. The same argument is also valid for the case

TABLE 1. Order of $|G|$ and lower bound for p when $p \nmid |S|$

S	$\lvert G\rvert$	Δ	$p \geq$
$G_2(r_1)$	$r_1^6\Phi_1^2\Phi_2^2\Phi_3\Phi_6$	$\{1,2,3,6\}$	5
${}^3D_4(r_1)$	$r_1^{12}\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12}$	$\{1,2,3,6,12\}$	5
$F_4(r_1)$	$r_1^{24}\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$	$\{1,2,3,4,6,8,12\}$	5
$E_6(r_1)$	$r_1^{36}\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	$\{1,2,3,4,5,6,8,9,12\}$	5
${}^2E_6(r_1)$	$r_1^{36}\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$	$\{1,2,3,4,6,8,10,12,18\}$	5
$E_7(r_1)$	$r_1^{63}\Phi_1^7\Phi_2^7\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$	$\{1,\dots,10,12,14,18\}$	7
$E_8(r_1)$	$r_1^{120}\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$	$\{1,\dots,10,12,14,15,18,20,24,30\}$	7
${}^2B_2(r_1)$	$r_1^2\Phi_1\Phi_4$	$\{1,4\}$	3
${}^2F_4(r_1)$	$r_1^{12}\Phi_1^2\Phi_2^2\Phi_4^2\Phi_6\Phi_{12}$	$\{1,2,4,6,12\}$	5
${}^2G_2(r_1)$	$r_1^3\Phi_1\Phi_2\Phi_6$	$\{1,2,6\}$	5

$S = {}^2G_2(r_1)$ or ${}^2F_4(r_1)$. If $S = E_6(r_1)$, then $p = 3$ and $e = 1$ or 2, or $p = 5$ and $e = 1$. In any case, $\Phi_e(r_1)$ is not the unique cyclotomic polynomial factor of the order polynomial of G which is divisible by p, a contradiction. A similar argument holds true for $S = {}^2E_6(r_1), E_7(r_1)$ or $E_8(r_1)$. This proves the claim $|C_G(\phi)|_q = 1$.

According to [9, Theorem 2.5.17], there is a Steinberg endomorphism σ of $\mathbf{G}$ such that $F \in \langle\sigma\rangle$ and σ induces the field automorphism ϕ of S which can also be viewed as a field automorphism of G. Hence the finite group $G_1 := \mathbf{G}^\sigma$ of fixed points is exactly the centralizer $C_G(\phi)$ of ϕ in G. In particular, $|G_1|_q = 1$.

Assume that $\sigma(s) = s^g$ for some nontrivial q-element s of G and $g \in G$. Recall that ${}^2B_2(r_1)$ will not occur as S. Thus, if $q = 3$, then $|C_G(\phi)|_3 \neq 1$ for all possible types of S, a contradiction. This implies $q \geq 5$ and s must be an element of S. By the Lang-Steinberg Theorem [20, Theorem 21.7], there is some $x \in \mathbf{G}$ such that $g^{-1} = \sigma(x)x^{-1}$. We have

$$\sigma(s^x) = \sigma(s)^{\sigma(x)} = (x^{-1}g) \cdot (g^{-1}sg) \cdot (g^{-1}x) = s^x.$$

Hence $s^x \in G_1$, and so $q \mid |G_1|$, a contradiction. Thus, σ and so ϕ does not fix the conjugacy class of some q-element of S in G, which finishes the proof. □

Proof of Theorem 1.3. According to [17, Propositions 3.2-3.3], among the sporadic simple groups, the alternating groups and the Tits simple group, which have a nilpotent Hall $\{p, q\}$-subgroup, are only J_1 with $\{p, q\} = \{3, 5\}$ and J_4 with $\{p, q\} = \{5, 7\}$, both of which imply $A = S$ and so do not occur in our situation. Thus S is a group of Lie type.

We first suppose that $pq \mid |S|$. Note that q is not the defining characteristic of S, by [17, Proposition 3.4]. Thus, by Proposition 3.1, $A = S\langle\phi\rangle$ for some field automorphism ϕ of S of order p^a and ϕ does not fix the conjugacy class of some q-element s of S in the subgroup of $\mathrm{Aut}(S)$ generated by the inner automorphisms and the diagonal automorphisms of S.

Let $\mathbf{G}$ be a simple simply-connected algebraic group and let F be a Frobenius endomorphism of $\mathbf{G}$ such that $S = \mathbf{G}^F/Z(\mathbf{G}^F)$. As mentioned in the second paragraph of the proof of Proposition 3.1, the primes p, q are odd and different from the defining characteristic of S, and any Hall $\{p, q\}$-subgroup of G is abelian and contained in some Sylow Φ_e-torus of $\mathbf{G}$, where $e = \mathrm{ord}_p(r_1) = \mathrm{ord}_q(r_1)$.

Let $(\mathbf{G}^*, F^*)$ be in duality with $(\mathbf{G}, F)$ (see [6, Section 4.3] for instance). Write $G^* := (\mathbf{G}^*)^{F^*}$ and $S^* := (G^*)'$ so that $|G| = |G^*|$ by [6, Proposition 4.4.4], and $S^* \cong S$ unless S is of type B_n or C_n. Moreover, it follows from [6, Corollary 4.4.2] that G^* also has an abelian Hall $\{p, q\}$-subgroup whose order is the same as for G.

The field automorphism ϕ of S induces a field automorphism ϕ^* of S^* of order p^a. According to [9, Theorem 2.5.17], there exists a Steinberg endomorphism σ^* of $\mathbf{G}^*$ such that $F^* = (\sigma^*)^{p^a}$ and σ^* induces ϕ^*. Since, by assumption, A does not have a nilpotent Hall $\{p, q\}$-subgroup, ϕ does not centralize any Sylow q-subgroup of S. Thus $|C_{\mathbf{G}^F}(\phi)|_q < |\mathbf{G}^F|_q$. Furthermore,

$$|C_{\mathbf{G}^F}(\phi)|_q = |C_{\mathbf{G}}(\sigma)|_q = |C_{\mathbf{G}^*}(\sigma^*)|_q = |C_{G^*}(\phi^*)|_q$$

and

$$|\mathbf{G}^F|_q = |G^*|_q.$$

Hence $|C_{G^*}(\phi^*)|_q < |G^*|_q$. Thus ϕ^* does not centralize any Sylow q-subgroup of G^*. Now by Proposition 3.1, ϕ^* does not fix the conjugacy class of some q-element s of S^* in G^*.

By [29, Proposition 7.2], the Lusztig series $\mathcal{E}(G, s)$ corresponding to s is not fixed by ϕ. On the other hand, by [12, Theorem 2.3], $B_q(G) \subseteq \coprod_t \mathcal{E}(G, t)$, and there is a character in $\mathcal{E}(G, t)$ which lies in the principal q-block of G, where t runs through all q-elements of G^* up to conjugation. Hence there is a character χ in $\mathcal{E}(G, s)$, which lies in the principal q-block of G and is not fixed by ϕ. Since $s \in S^* = [G^*, G^*]$, we have $\mathrm{ker}\chi = Z(G)$ by [25, Lemma 4.4], so χ is indeed a character of S. Thus the theorem holds in this case.

We now suppose that $p \nmid |S|$. Then S is of Lie type other than the Tits simple group, and we may assume that $A = S\langle\phi\rangle$ for some field automorphism ϕ of S, by [9, Theorem 7.1.2]. We may further assume that A does not have nilpotent Hall $\{p, q\}$-subgroups. If $q \neq r$, we may similarly argue as above with Proposition 3.1. Finally, let $q = r$. In this case, all irreducible characters of S apart from the Steinberg character of S lie in $B_q(S)$. If A acts trivially on $B_q(S)$, then A fixes all irreducible characters of S, since the Steinberg character of S is invariant under the conjugate action of $\mathrm{Aut}(S)$. Thus, by Brauer's permutation lemma, ϕ fixes all conjugacy classes of S, a contradiction to [7, Theorem C]. This finishes the proof. □

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