Involutions, cohomology and metabolic spaces

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Abstract We prove that over a field of characteristic two the involution module of a finite group allows a natural splitting into a direct sum of cohomology modules.

1 Introduction and main results

Throughout this paper let K be a perfect field of characteristic 2 and let G be a finite group. We may assume that K is a splitting field for G although some of the results are valid without this assumption. All modules are assumed to be right-modules of finite K-dimension over group algebras KG. By $\mathrm{Inv}(G)$ we denote the set of involutions in G. Furthermore, for a KG-module V, let V^* be the dual module of V, $V^{(2)}$ the Frobenius twist of V, and let $\Lambda^2(V) = V \wedge V$ be the wedge product of V by itself. Finally, we denote by $\mathrm{H}^1(G, V)$ the first cohomology group of G with coefficients in V and, for a 2-block B, by $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$ the set of irreducible classical resp. 2-Brauer characters belonging to B. If 1 has the meaning of a character then it denotes the trivial character.

The group G acts on the group algebra KG by conjugation and the vector space $K\,\mathrm{Inv}(G)$ generated by the involutions is a submodule of KG via this action. The space $K\,\mathrm{Inv}(G)$, usually called the involution module, has been studied intensively by J. Murray in [5], [6] and [7]. In this paper we relate the involution module $K\,\mathrm{Inv}(G)$ to cohomology modules. Observe that the conjugation action induces a G-module structure of $\mathrm{H}^1(G, \wedge^2(B))$ for any two-sided ideal B of KG (see Lemma 2.2 below). We prove the following results.

Theorem A. *If $\mathcal{B}^\circ$ denotes the set of self-dual blocks of KG then*

$$K\,\mathrm{Inv}(G) \cong \mathrm{H}^1(G, \Lambda^2(KG)) \cong \oplus_{B \in \mathcal{B}^\circ} \mathrm{H}^1(G, \Lambda^2(B))$$

as KG-modules. In particular,

$$|\,\mathrm{Inv}(G)| = \dim_K\, \mathrm{H}^1(G, \Lambda^2(KG)) = \sum_{B \in \mathcal{B}^\circ} \dim_K\, \mathrm{H}^1(G, \Lambda^2(B)).$$

Moreover, all composition factors of $\mathrm{H}^1(G, \Lambda^2(B))$ belong to the block $B^{(2)}$.

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Theorem B. *If B is a self-dual block of defect zero with simple module V then as KG-modules we have* $\mathrm{H}^1(G, \Lambda^2(B)) \cong V^{(2)}$.

As a vector space the involution module $K\,\mathrm{Inv}(G)$ has a finer natural splitting than given in Theorem A, namely

$$K\,\mathrm{Inv}(G) \cong \oplus_{\substack{\varphi\in\mathrm{IBr}(B)\\ B\in\mathcal{B}^\circ}} \varphi(1)\,\mathrm{H}^1(G, \Lambda^2(P_\varphi)) \tag{1}$$

where P_φ denotes the projective cover of the module afforded by the irreducible Brauer character φ. Let Φ_φ denote the classical character associated to P_φ. For a generalized character $\Phi = \sum_{\chi\in\mathrm{Irr}(G)} e_{\phi,\chi}\chi$ with $e_{\phi,\chi} \in \mathbb{Z}$ we define the Frobenius-Schur indicator by

$$\nu(\Phi) = \sum_{\chi\in\mathrm{Irr}(G)} e_{\Phi,\chi}\nu(\chi)$$

where $\nu(\chi)$ is the Frobenius-Schur indicator of the irreducible character χ (see [4], section 4).

Theorem C. *We have*

$$dim_K\, H^1(G, \Lambda^2(P_\varphi)) \quad = \quad \begin{cases} \nu(\Phi_\varphi) & \text{if } 1 \neq \varphi \\ \nu(\Phi_\varphi) - 1 & \text{if } 1 = \varphi. \end{cases}$$

Thus, according to a result of Murray [5]*,* $dim_K\, H^1(G, \Lambda^2(P_\varphi))$ *is the multiplicity of* φ *as a constituent in the Brauer character of the involution module* $K\,\mathrm{Inv}(G)$*.*

In order to state a geomerical meaning of $\mathrm{H}^1(G, \Lambda^2(P_\varphi))$ for an indecomposable projective KG-module we define metabolic G-spaces. For a KG-module V, let $M(V) = V^* \oplus V$. We call $(M(V), b)$ a *metabolic G-space* if b is a non-degenerate G-invariant bilinear symplectic form on $M(V)$ which is totally zero on V^*. Observe that $V^* \oplus V$ carries at least one such form b, namely

$$b(f_1 + v_1, f_2 + v_2) = f_1(v_2) + f_2(v_1)$$

for $f_i \in V^*$ and $v_i \in V$. With this notation we have

Theorem D. *Let P be a projective indecomposable KG-module. Then there is a one to one correspondence between* $\mathrm{H}^1(G, \Lambda^2(P^*))$ *and the set of isometry classes of metabolic G-spaces* $(\mathrm{M}(P), b)$*.*

2 G-invariant forms and involutions

For the reader's convenience we recall some facts about G-invariant forms. For more details we refer to [9] and ([3], Chap VII, section 8).

Let F be any field, V an FG-module, and let $V^* \otimes_F V^*, \mathrm{S}^2(V^*)$, $S(V^*)$, $\Lambda^2(V^*)$ denote the space of bilinear, quadratic, symmetric resp. symplectic forms on V. Since we are mainly interested in G-invariant forms we have to consider the fixed point spaces

$$(V^*\otimes_F V^*)^G,\ \mathrm{S}^2(V^*)^G,\ \mathrm{S}(V^*)^G \text{ and } \Lambda^2(V^*)^G$$

which are the spaces of G-invariant bilinear, quadratic, symmetric resp. symplectic forms on V. Clearly, if $V\cong V^*$ then $(V^*\otimes_F V^*)^G\cong \mathrm{End}_{FG}(V)$.

From the space $V^*\otimes_F V^*$ of all bilinear forms on V there is a natural map

$$\delta : V^*\otimes_F V^* \to \mathrm{S}^2(V^*)$$

which associates to a bilinear form b the quadratic form q defined by $q(v)=b(v,v)$ for $v\in V$. This yields the short exact sequence

$$0\to \Lambda^2(V^*)\to V^*\otimes_F V^* \xrightarrow{\delta} \mathrm{S}^2(V^*)\to 0. \tag{2}$$

We also have the short exact sequence

$$0\to \mathrm{S}(V^*)\to V^*\otimes_F V^* \xrightarrow{\sigma} \Lambda^2(V^*)\to 0 \tag{3}$$

where σ is the anti-symmetrization, i.e., $b\sigma=b-\bar{b}$ for a bilinear form b where $\bar{b}$ is defined by

$$\bar{b}(v,v')=b(v',v) \text{ for } v,v'\in V.$$

If $\mathrm{Char}\,F\neq 2$ then $\mathrm{S}^2(V^*)\cong S(V^*)$ and the sequences (2) and (3) split. Furthermore,

$$(V^*\otimes_F V^*)^G\cong \Lambda^2(V^*)^G\oplus \mathrm{S}^2(V^*)^G.$$

From now on we assume that $\mathrm{Char}\,F=2$, i.e., $F=K$. The polarization map

$$\rho : \mathrm{S}^2(V^*)\to \Lambda^2(V^*)$$

is defined by $q\rho=b$ with

$$b(v,v')=q(v+v')+q(v)+q(v') \qquad (\text{for } v,v'\in V)$$

where q is a quadratic form on V. If $V^{(2)}$ denotes the Frobenius twist of the KG-module V then via the polarization we obtain the following short exact sequence

$$0\to V^{*(2)}\to \mathrm{S}^2(V^*)\xrightarrow{\rho} \Lambda^2(V^*)\to 0. \tag{4}$$

Taking G-fixed points in (4) and (2) we get the long exact sequences

$$0\to (V^{*(2)})^G\to \mathrm{S}^2(V^*)^G\to \Lambda^2(V^*)^G\to \mathrm{H}^1(G,V^{(2)*})\to \quad .... \tag{5}$$

and

$$0\to \Lambda^2(V^*)^G\to (V^*\otimes_K V^*)^G\to \mathrm{S}^2(V^*)^G\to \mathrm{H}^1(G,\Lambda^2(V^*))\to \mathrm{H}^1(G,V^*\otimes_K V^*) \quad .... \tag{6}$$

Next observe that the cohomology vanishes on projective modules. Thus, if we specialize the sequences (5) and (6) to projective modules P, we obtain the exact sequences

$$0 \to (P^{*(2)})^G \to \mathrm{S}^2(P^*)^G \xrightarrow{\rho} \Lambda^2(P^*)^G \to 0 \tag{7}$$

and

$$0 \to \Lambda^2(P^*)^G \to (P^* \otimes_K P^*)^G \xrightarrow{\delta} \mathrm{S}^2(P^*)^G \to \mathrm{H}^1(G, \Lambda^2(P^*)) \to 0. \tag{8}$$

Moreover, there is a map from $\mathrm{S}(P^*)$ onto $P^{*(2)}$ with kernel $\Lambda^2(P^*)$ defined by

$$b \mapsto \alpha$$

where $x\alpha = b(x, x)$ for $x \in P$. Since P is projective we have

$$\mathrm{S}(P^*) \cong \Lambda^2(P^*) \oplus P^{*(2)}. \tag{9}$$

Thus taking fixed points in (3) leads to the exact sequence

$$0 \to \mathrm{S}(P^*)^G \to (P^* \otimes_K P^*)^G \xrightarrow{\sigma} \Lambda^2(P^*)^G \to \mathrm{H}^1(G, \mathrm{S}(P^*)) = \mathrm{H}^1(G, \Lambda^2(P^*)) \to 0. \tag{10}$$

In order to prove Theorem A we need the following well-known fact.

Lemma 2.1 *Let V be a KG-module with a non-degenerate G-invariant bilinear form b. Note that the existence of b is equivalent to $V \cong V^*$ as KG-modules. Then any G-invariant bilinear form b' on V is of the form $b'(\cdot\,,\cdot) = b(\cdot\,\alpha\,,\cdot)$ for some $\alpha \in \mathrm{End}_{KG}(V)$.*

Next we specialize to the case that $V = B$ is a two-sided ideal of KG. If B^{op} denotes the opposite algebra of B and 1_B the identity of B then the map

$$\mathrm{End}_{KG}(B) \ni \alpha \mapsto 1_B\alpha \in B$$

defines an algebra isomorphism between $\mathrm{End}_{KG}(B)$ and B^{op} since

$$a\alpha = (1_B a)\alpha = (1_B\alpha)a$$

for $a \in B$. In particular, α acts on B via left multiplication by $1_B\alpha$. Furthermore, on the group algebra KG we have the canonical symmetric non-degenerate G-invariant bilinear form b defined by

$$b(g, h) = \delta_{g,h} \quad \text{for } g, h \in G.$$

If we assume that B is self-dual, i.e. $B = B^*$, then the restriction of b to the subalgebra B remains non-degenerate. Thus, for $B = B^*$,

$$\mathrm{End}_{KG}(B) \cong \{b_a(\cdot\,,\cdot) = b(a\cdot\,,\cdot) \mid a \in B\}$$

is the space of all G-invariant bilinear forms of B. Moreover, G acts on this space by

$$b_a \circ g = b_{a^g}$$

for $a \in B$ and $g \in G$. Via this action, we may identify $\mathrm{End}_{KG}(B) = (B \otimes_K B)^G$ with B^{conj} where B^{conj} denotes the G-module with B as underlying vector space and conjugation action by G.

Let $a \mapsto \hat{a}$ denotes the antialgebra automorphism of KG given by $g \mapsto g^{-1}$ and let $\mathrm{Tr}(a) = b(a,1)$ be the coefficient of a at 1. Then the G-invariant symplectic forms on $B = B^*$, i.e. the elements in $\Lambda^2(B)^G$, can be described in the following way.

Lemma 2.2 *For a self-dual block or more generally a self-dual two-sided ideal B of KG we have*

a) $\Lambda^2(B)^G = \{a \in B \mid a = \hat{a},\ \mathrm{Tr}(a) = 0\} \leq B^{conj}$ *as KG-modules.*

b) $(B \otimes_K B)^G \sigma_B = \mathrm{Im}\,\sigma_B = \{a + \hat{a} \mid a \in B\} \leq \Lambda^2(B)^G$ *as KG-modules.*

c) $\Lambda^2(B)^G/\mathrm{Im}\,\sigma_B \cong \mathrm{H}^1(G, \Lambda^2(B))$*. In particular* $\mathrm{H}^1(G, \Lambda^2(B))$ *is a KG-module.*

Proof: a) Clearly, if $a = \hat{a} \in B$ with $\mathrm{Tr}(a) = 0$ then b_a defines a G-invariant symplectic form on B. Conversely, let $a = \sum_{g\in g} a_g g \in B$ such that b_a is a symplectic form on B. Thus, for $g \in G$, we have

$$a_g = b(a, g) = b_a(1, g) = b_a(g, 1) = b(ag, 1) = a_{g^{-1}}$$

and

$$0 = b_a(1, 1) = b(a, 1) = a_1.$$

b) Since $\overline{b_a} = b_{\hat{a}}$ and $\widehat{a^g} = \hat{a}^g$ for $a \in KG$ and $g \in G$ the anti-symmetrization σ is a KG-homomorphism. The claim now follows as $b_g\sigma = b_g + b_{g^{-1}}$ for $g \in G$.
c) The isomorphism as vector spaces is an immediate consequence of (10). The module structure of $\Lambda^2(B)^G/\mathrm{Im}\,\sigma_B$ follows from a) and b).

□

Proof of the first part of Theorem A.
Clearly, $KG \cong KG^*$. Thus, by Lemma 2.2 with $B = KG$, we have

$$\Lambda^2(KG)^G = \{ \sum_{t\in G,\, t^2=1} a_t t + \sum_{g\in G,\, g^2\neq 1} a_g(g + g^{-1}) \in KG\}$$

and

$$\mathrm{Im}\,\sigma = \{ \sum_{g\in G,\, g^2\neq 1} a_g(g + g^{-1}) \in KG\},$$

where G acts on both spaces via conjugation. Therefore, as KG-modules, we have

$$K\,\mathrm{Inv}(G) \cong \Lambda^2(KG)^G/\mathrm{Im}\,\sigma \cong \mathrm{H}^1(G, \Lambda^2(KG)),$$

which proves the first isomorphism of the theorem.

In order to see the second isomorphism, note that for any two KG-modules, say V and W, there is a natural direct splitting

$$\Lambda^2(V \oplus W) \cong \Lambda^2(V) \oplus \Lambda^2(W) \oplus V \otimes_K W.$$

If V is projective then $V \otimes_K W$ is projective as well and the cohomology with values in it vanishes. Thus

$$\mathrm{H}^1(G, \Lambda^2(KG)) \cong \oplus_{B \in \mathcal{B}} \mathrm{H}^1(G, \Lambda^2(B))$$

where $\mathcal{B}$ denotes the set of all blocks. If B is not self-dual then

$$\Lambda^2(B)^G \subseteq (B \otimes_K B)^G \cong \mathrm{Hom}_{KG}(B, B^*) = 0.$$

Thus (10) yields $\mathrm{H}^1(G, \Lambda^2(B)) = 0$ for $B \neq B^*$ and we obtain

$$\mathrm{H}^1(G, \Lambda^2(KG)) = \oplus_{B \in \mathcal{B}^\circ} \dim_K \mathrm{H}^1(G, \Lambda^2(B)).$$

□

3 The block components of the involution module

In this section we analyze the module $\mathrm{H}^1(G, \Lambda^2(B))$ where B is a two-sided self-dual ideal of KG. Recall that, by Theorem A, the module $\mathrm{H}^1(G, \Lambda^2(B))$ is a direct summand of the involution module. Denote by $B_{\mathbb{C}}$ the two-sided ideal of the group algebra $\mathbb{C}G$ associated to B. Furthermore, we use $B_{\mathbb{C}}^{\mathrm{conj}}$ to refer to this module with the conjugation action of G. Finally, we write $\chi \in \mathrm{Irr}(B)$ if χ is an irreducible character belonging to a block occuring in the decomposition of the two-sided ideal B in blocks.

For an irreducible complex character χ let

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

denote the Frobenius-Schur indicator. Recall that if V is a $\mathbb{C}G$-module afforded by χ then $\bar{\chi}$ is the character of V^* and

$$\nu(\chi) = \begin{cases} 0 & \text{if } \chi \neq \bar{\chi} \\ 1 & \text{if } \chi = \bar{\chi} \text{ and } V \text{ can be realised over } \mathbb{R} \\ -1 & \text{if } \chi = \bar{\chi} \text{ and } V \text{ can not be realised over } \mathbb{R}. \end{cases}$$

(see [4], section 4) or equivalently

$$\nu(\chi) = \begin{cases} 0 & \text{if } V \text{ does not carry a } G\text{-invariant bilinear form} \neq 0 \\ 1 & \text{if } V \text{ admits a } G\text{-invariant symmetric form} \neq 0 \\ -1 & \text{if } V \text{ admits a } G\text{-invariant symplectic form} \neq 0. \end{cases}$$

Thus we have

$$\nu(\chi) = \dim_{\mathbb{C}} S(V)^G - \dim_{\mathbb{C}} \Lambda^2(V)^G.$$

Lemma 3.1 *The character of $B_{\mathbb{C}}^{conj}$ is $\sum_{\chi \in \mathrm{Irr}(B)} \chi\overline{\chi}$.*

Proof: It suffices to prove that the indecomposable two-sided ideal A of $\mathbb{C}G$ associated to $\chi \in \mathrm{Irr}(G)$ affords the character $\chi\overline{\chi}$ if G acts by conjugation on A. We may consider A as a $G \times G$-module with the action

$$a \cdot (g,h) = g^{-1}ah.$$

If we put $G_1 = G\times 1$ and $G_2 = 1\times G$ then the character of G_2 on the restriction $A|_{G_2}$ is $\chi(1)\chi$ and the character of G_1 on $A|_{G_1}$ is $\chi(1)\bar{\chi}$. Clearly, A is irreducible as $G \times G$-module and therefore the character of $G \times G$ on A is $\bar{\chi} \times \chi$. Finally, the character of the G-conjugation action on A is just the restriction to the diagonal subgroup $\{(g,g) \mid g \in G\}$ of $G \times G$. $\square$

Lemma 3.2 *Let $e \in \mathbb{C}G$ be the central idempotent associated to $\chi \in \mathrm{Irr}(G)$. If ψ_χ denotes the character of the G-conjugation action on $\{a - \hat{a} \mid a \in e\mathbb{C}G\}$ then*

$$\psi_\chi = \begin{cases} \Lambda^2\chi & \text{if } \nu(\chi) = 1 \\ S\chi & \text{if } \nu(\chi) = -1 \\ \chi\bar{\chi} & \text{if } \nu(\chi) = 0 \end{cases}$$

where $\Lambda^2\chi$ resp. $S\chi$ are the characters of the space of antisymmetric resp. symmetric tensors of the module afforded by $\chi = \overline{\chi}$.

Proof: First we consider the case $\nu(\chi) = 0$, i.e., $\bar{\chi} \neq \chi$. In this case $e\mathbb{C}G \neq \hat{e}\mathbb{C}G$, but, by Lemma 3.1, both modules are isomorphic as G-modules with the conjugation action and the character of any of them is $\chi\overline{\chi}$. Next we decompose

$$e\mathbb{C}G \oplus \hat{e}\mathbb{C}G = \{a - \hat{a} \mid a \in e\mathbb{C}G\} \oplus \{a + \hat{a} \mid a \in e\mathbb{C}G\}$$

and observe that $\{a - \hat{a} \mid a \in e\mathbb{C}G\}$ and $\{a + \hat{a} \mid a \in e\mathbb{C}G\}$ are G-isomorphic via

$$a - \hat{a} \mapsto a + \hat{a}.$$

Thus $2\chi\bar{\chi} = 2\psi_\chi$, and we are done in this case.

From now on we assume that $\chi = \bar{\chi}$. Let V be a $\mathbb{C}G$-module affording the character χ. We know already that

$$\{a - \hat{a} \mid a \in e\mathbb{C}G\} \oplus \{a + \hat{a} \mid a \in e\mathbb{C}G\} = e\mathbb{C}G^{\mathrm{conj}} \cong V \otimes_{\mathbb{C}} V = S(V) \oplus \Lambda^2(V)$$

where G acts diagonally on $V \otimes_{\mathbb{C}} V$. We are going to construct explicitely an isomorphism α from $e\mathbb{C}G^{\mathrm{conj}}$ onto $V \otimes_{\mathbb{C}} V$ which allows us to prove the claim. First, let ρ denote the representation of G on $V = \mathbb{C}^n$. Then

$$\begin{aligned} V \otimes_{\mathbb{C}} V = \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n &\xrightarrow{\varrho} (\mathbb{C})_n \\ v \otimes w &\mapsto M = v^t \otimes w \end{aligned}$$

defines a G-isomorphism if G acts on $(\mathbb{C})_n$ via

$$M \mapsto \rho(g)^t M \rho(g).$$

Since $(1_G, \chi^2) = (\chi, \chi) = 1$ there exists a matrix $0 \neq P \in (\mathbb{C})_n$ such that $P = \rho(g)^t P \rho(g)$ for any $g \in G$. Moreover, via ϱ we have

$$\Lambda^2(V) \overset{\varrho}{\cong} \{M \mid M \in (\mathbb{C})_n,\ M^t = -M\},$$

$$S(V) \overset{\varrho}{\cong} \{M \mid M \in (\mathbb{C})_n,\ M^t = M\}.$$

Now as $\nu(\chi) = (1_G, S(\chi) - \Lambda^2(\chi))$ we deduce that the matrix P satisfies the condition

$$\begin{array}{ll} P^t = P & \text{if } \nu(\chi) = 1, \\ P^t = -P & \text{if } \nu(\chi) = -1, \end{array}$$

i.e., $P^t = \nu(\chi)P$. Clearly, P is regular since P defines an isomorphism from V^* to V.

Next, we consider $(\mathbb{C})_n^{\mathrm{conj}}$, i.e., the same vector space $(\mathbb{C})_n$ as before but with the conjugation action given by

$$M \mapsto \rho(g)^{-1} M \rho(g).$$

Via ρ we may identify any $a \in e\mathbb{C}G$ with the matrix $\rho(a) \in (\mathbb{C})_n$, that is, ρ yields an G-isomorphism between $e\mathbb{C}G^{\mathrm{conj}}$ and $(\mathbb{C})_n^{\mathrm{conj}}$. Let

$$\begin{array}{rl} \alpha:\ e\mathbb{C}G^{\mathrm{conj}} \xrightarrow{\rho} (\mathbb{C})_n^{\mathrm{conj}} & \to (\mathbb{C})_n \xrightarrow{\varrho^{-1}} V \otimes_{\mathbb{C}} V \\ M & \mapsto PM. \end{array}$$

Note that the map α is a G-homomorphism since

$$P\rho(g)^{-1} M \rho(g) = \rho(g)^t P M \rho(g).$$

Moreover, for any $g \in G$ we have

$$\begin{aligned} (P\rho(g) - P\rho(g)^{-1})^t &= \rho(g)^t P^t - (\rho(g)^{-1})^t P^t = \\ &= \nu(\chi) P \rho(g)^{-1} - \nu(\chi) P \rho(g) = -\nu(\chi)(P\rho(g) - P\rho(g)^{-1}), \end{aligned}$$

and analogously

$$(P\rho(g) + P\rho(g)^{-1})^t = \nu(\chi)(P\rho(g) + P\rho(g)^{-1}).$$

This implies that via α

$$\{a - \hat{a} \mid a \in e\mathbb{C}G\} \overset{\alpha}{\cong} \begin{cases} \Lambda^2(V) & \text{if } \nu(\chi) = 1 \\ S(V) & \text{if } \nu(\chi) = -1, \end{cases}$$

which finishes the proof.

□

In the following results we denote by χ^0 the Brauer character associated to the character χ, i.e., χ^0 is the restriction of χ on the set of p-regular elements.

Lemma 3.3 *Let* $\Lambda_B = \{a - \hat{a} \mid a \in B_{\mathbb{C}}\} \leq B_{\mathbb{C}}^{conj}$.

a) The character of the $\mathbb{C}G$*-module* Λ_B *is*

$$\psi = \sum_{\substack{\chi\in\mathrm{Irr}(B)\\ \nu(\chi)=1}} \wedge^2\chi \quad + \sum_{\substack{\chi\in\mathrm{Irr}(B)\\ \nu(\chi)=-1}} S\chi \quad + \frac{1}{2}\sum_{\substack{\chi\in\mathrm{Irr}(B)\\ \nu(\chi)=0}} \chi\overline{\chi}.$$

b) ψ^0 *is the Brauer character of* $\mathrm{Im}\,\sigma_B$.

Proof: a) This follows directly from Lemma 3.2. Note that for each $\chi \neq \bar{\chi} \in \mathrm{Irr}G$, the character $\chi\bar{\chi}$ appears twice in the last sum.
b) If we put

$$\Lambda = \{a - \hat{a} \mid a \in \mathbb{C}G\} = \langle g - g^{-1} \mid g \in G,\, g^2 \neq 1\rangle$$

then there is a G-module decomposition

$$\Lambda = \Lambda_B \oplus \Lambda_C$$

for certain two-sided ideal C of $\mathbb{C}G$ such that $\mathbb{C}G = B_{\mathbb{C}} \oplus C$.

By Lemma 2.2 b), we see that ψ^0 is the Brauer character of the image of σ_B. □

The next result finishes the proof of Theorem A.

Proposition 3.4 *All composition factors of* $\mathrm{H}^1(G, \Lambda^2(B))$ *lie in* $B^{(2)}$. *Moreover, the Brauer character of* $\mathrm{H}^1(G, \Lambda^2(B))$ *is given by*

$$\sum_{1\neq\chi\in\mathrm{Irr}(B)} \nu(\chi)(\chi^0)^{(2)}.$$

Proof: By Lemma 3.1, the Brauer character of $B_{\mathbb{C}}^{\mathrm{conj}}$ is $\sum_{\chi\in\mathrm{Irr}(B)}(\chi\bar{\chi})^0$. On the other hand, by Lemma 2.2 and the exact sequences (7) and (8), this character is

$$2\psi^0 + \beta$$

where ψ^0 is the Brauer character of $\mathrm{Im}\,\sigma$ and β the Brauer character of $\mathrm{H}^1(G, \Lambda^2(B)) \oplus (B^{(2)})^G$. Thus, by Lemma 3.3, we have

$$\begin{aligned}\sum_{\chi\in\mathrm{Irr}(B)} (\chi\overline{\chi})^0 \quad &= \sum_{\substack{\chi\in\mathrm{Irr}(B)\\ \nu(\chi)\neq 0}} [(\wedge^2\chi)^0 + (S\chi)^0] \;+ \sum_{\substack{\chi\in\mathrm{Irr}(B)\\ \nu(\chi)=0}} (\chi\overline{\chi})^0 \\ &= \beta + 2\sum_{\substack{\chi\in\mathrm{Irr}(B)\\ \nu(\chi)=1}} (\wedge^2\chi)^0 + 2\sum_{\substack{\chi\in\mathrm{Irr}(B)\\ \nu(\chi)=-1}} (S\chi)^0 + \sum_{\substack{\chi\in\mathrm{Irr}(B)\\ \nu(\chi)=0}} (\chi\bar{\chi})^0.\end{aligned}$$

Therefore it follows

$$\beta = \sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)(S\chi - \wedge^2\chi)^0 = \sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)(\chi^0)^{(2)}.$$

□

The statement in Proposition 3.4 implies in particular Corollary 4 of [5].

Example 3.5 Let $G = D_{10}$ be a dihedral group of order 10. Then G has three irreducible KG-modules, namely the trivial module K, V_1 and V_2 where V_1 and V_2 belong to self-dual blocks B_1 resp. B_2 both of defect zero. One easily checks that $V_1^{(2)} = V_2$, i.e., $B_1^{(2)} = B_2$.

Proof of Theorem B. Assume that $B = B^*$ has defect zero and let χ be the only irreducible complex character in B. Then $\nu(\chi) = 1$ and χ^0 is the only irreducible Brauer character belonging to B. Let V be the irreducible KG-module affording χ^0. Then Proposition 3.4 implies $H^1(G, \Lambda^2(B)) \cong V^{(2)}$. □

We would like to mention here that, due to a result of Murray [6], the module $\mathrm{H}^1(G, \Lambda^2(B))$ does not contain any projective indecomposable direct summand unless the block B is of defect zero.

4 Dimensions of the cohomology groups

For a generalized character $\Phi = \sum_{\chi \in \mathrm{Irr}(G)} e_{\phi,\chi}\chi$ with $e_{\phi,\chi} \in \mathbb{Z}$ we define

$$\nu(\Phi) = \sum_{\chi \in \mathrm{Irr}(G)} e_{\Phi,\chi}\nu(\chi).$$

Furthermore, for $\chi \in \mathrm{Irr}(G)$ and $\varphi \in \mathrm{IBr}(G)$, we denote by $d_{\chi,\varphi}$ the corresponding decomposition number.

Let $\chi \in \mathrm{Irr}(G)$ and let V be a $\mathbb{C}G$-module affording the character χ. Then, as we have seen above,

$$\nu(\chi) = \dim_{\mathbb{C}} S(V)^G - \dim_{\mathbb{C}} \Lambda^2(V)^G. \tag{11}$$

Lemma 4.1 *Formula* (11) *holds true for any character* χ (*not necessarily irreducible*).

Proof: We proceed by induction on the composition length of V afforded by the character χ. Let $\chi = \chi_1 + \chi_2$ with corresponding modules $V = V_1 \oplus V_2$. Then

$$S(V_1 \oplus V_2)^G = S(V_1)^G \oplus S(V_2)^G \oplus (V_1 \otimes_{\mathbb{C}} V_2)^G,$$

$$\Lambda^2(V_1 \oplus V_2)^G = \Lambda^2(V_1)^G \oplus \Lambda^2(V_2)^G \oplus (V_1 \otimes_{\mathbb{C}} V_2)^G$$

which implies

$$\begin{aligned}
\nu(\chi_1) + \nu(\chi_2) &= \dim_{\mathbb{C}} S(V_1)^G - \dim_{\mathbb{C}} \Lambda^2(V_1)^G + \dim_{\mathbb{C}} S(V_2)^G - \dim_{\mathbb{C}} \Lambda^2(V_2)^G \\
&= \dim_{\mathbb{C}} S(V_1 \oplus V_2)^G - \dim_{\mathbb{C}} \Lambda^2(V_1 \oplus V_2)^G. \\
&= \nu(\chi).
\end{aligned}$$

□

Let (F, R, K) be a 2-modular system. So R is a complete discrete valuation ring with fraction field F, unique maximal ideal $\wp$ and residue field $R/\wp = K$ of characteristic 2.

Lemma 4.2 *Let P be a projective indecomposable KG-module and let Q be its lift to F. If $S(Q)^G$ and $S(P)^G$ denote the spaces of G-invariant bilinear symmetric forms on Q resp. P then*

$$dim_F\,\mathrm{S}(Q)^G \leq dim_K\,\mathrm{S}(P)^G.$$

Proof: Let P_R be the RG-module such that $P_R \otimes_R F = Q$ and $P_R/P_R\wp = P$. Consider the R-module of bilinear symmetric G-invariant forms on P_R, i.e.,

$$\mathrm{S}(P_R)^G = \{b : P_R \times P_R \to R \mid b \text{ is } R\text{-bilinear, symmetric and } G\text{-invariant}\}.$$

Obviously, $\mathrm{S}(P_R)^G \otimes_R F \subseteq \mathrm{S}(Q)^G$. Since each form in $\mathrm{S}(Q)^G$ can be lifted to a form in $\mathrm{S}(P_R)^G$ by multiplying with a suitable $a \in F$ we even have

$$\mathrm{S}(P_R)^G \otimes_R F = \mathrm{S}(Q)^G.$$

This implies that $\mathrm{S}(P_R)^G$ is finitely generated. Moreover R is a principal ideal domain and $\mathrm{S}(P_R)^G$ is torsion free. Thus $\mathrm{S}(P_R)^G$ is a free R-module of rank equal to $\dim_F \mathrm{S}(Q)^G$. Clearly, the natural epimorphisms $P_R \twoheadrightarrow P$ and $R \twoheadrightarrow K$ induce a well-defined homomorphism

$$\mathrm{S}(P_R)^G \to \mathrm{S}(P)^G.$$

The kernel of this map is the space of all forms in $\mathrm{S}(P_R)^G$ which take values in $\wp$. Since the ideal $\wp$ is principal this kernel is precisely $\mathrm{S}(P_R)^G\wp$. Therefore

$$\dim_F \mathrm{S}(Q)^G = \operatorname{rank} \mathrm{S}(P_R)^G = \dim_K \mathrm{S}(P_R)^G / \mathrm{S}(P_R)^G\wp \leq \dim_K \mathrm{S}(P)^G$$

□

The proof of Theorem C shows that in Lemma 4.2 actually equality holds.

Proof of Theorem C. For $\varphi \in \mathrm{IBr}(G)$, let $P = P_\varphi$ be the projective indecomposable KG-module associated to φ and let Q be its lift to F. We denote by Φ_φ the ordinary character associated to Q.

Recall that the Cartan invariant $c_{\varphi\overline{\varphi}}$ can be computed as

$$c_{\varphi\overline{\varphi}} = \sum_{\chi} d_{\chi\varphi} d_{\chi\bar{\varphi}} = \dim_F\ \mathrm{Hom}_{FG}(Q^*, Q) = \dim_F (Q \otimes_F Q)^G = \dim_F \mathrm{S}(Q)^G + \dim_F \Lambda^2(Q)^G$$

Furthermore, the exact sequence in (9) implies that

$$\dim_K \mathrm{S}(P)^G = \dim_K \Lambda^2(P)^G + \epsilon$$

where $\epsilon = 1$ if $P = P_1$ (the projective indecomposable module with trivial head) and $\epsilon = 0$ otherwise. Therefore counting dimensions in the exact sequence in (10) yields

$$\begin{aligned}
\dim_K H^1(G, \Lambda^2(P)) &= \dim_K\ \mathrm{S}(P)^G + \dim_K \Lambda^2(P)^G - c_{\varphi\bar{\varphi}} \\
&= \dim_K\ \mathrm{S}(P)^G + \dim_K \Lambda^2(P)^G - \dim_F\ \mathrm{S}(Q)^G - \dim_F \Lambda^2(Q)^G \\
&= 2\dim_K\ \mathrm{S}(P)^G - \dim_F\ \mathrm{S}(Q)^G - \dim_F \Lambda^2(Q)^G - \epsilon \\
&\geq \dim_F\ \mathrm{S}(Q)^G - \dim_F \Lambda^2(Q)^G - \epsilon \\
&= \nu(\Phi_\varphi) - \epsilon
\end{aligned}$$

where the inequality is a consequence of Lemma 4.2 and the last equality follows from Lemma 4.1.

On the other hand, by ([4], Corollary 4.6), we have

$$1 + |\operatorname{Inv}(G)| = \sum_{\chi \in \operatorname{Irr}(G)} \chi(1)\nu(\chi).$$

Thus, Theorem A and the bound above imply

$$\begin{array}{rcl}\sum_{\chi\in\operatorname{Irr}(G)}\chi(1)\nu(\chi) & = & 1+|\operatorname{Inv}(G)| \\ & = & 1+\sum_{\varphi\in\operatorname{IBr}(G)}\varphi(1)\dim_K H^1(G,\Lambda^2(P_\varphi)) \\ & \geq & \sum_{\varphi\in\operatorname{IBr}(G)}\varphi(1)\nu(\Phi_\varphi) \\ & = & \sum_{\varphi\in\operatorname{IBr}(G)}\varphi(1)\sum_{\chi\in\operatorname{Irr}(G)}d_{\chi\varphi}\nu(\chi) \\ & = & \sum_{\chi\in\operatorname{Irr}(G)}\{\sum_{\varphi\in\operatorname{IBr}(G)}\varphi(1)d_{\chi\varphi}\}\,\nu(\chi) \\ & = & \sum_{\chi\in\operatorname{Irr}(G)}\chi(1)\nu(\chi).\end{array}$$

By Lemma 3 of [5], we know that $\nu(\Phi_\varphi) - \epsilon$ is the multiplicity of $\varphi \in \operatorname{IBr}(G)$ in the Brauer character of the involution module $K\operatorname{Inv}(G)$. In particular $\nu(\Phi_\varphi) - \epsilon \geq 0$. Thus the inequalities $\dim H^1(G, \Lambda^2(P_\varphi)) \geq \nu(\Phi_\varphi) - \epsilon$ are in fact all equalities and the proof is complete. □

The proof of Theorem C implies that $\dim_K H^1(G, \Lambda^2(P)) = \nu(\Phi) - m$ for any projective KG-module P (not necesarily indecomposable) with classical character Φ where m denotes the multiplicity of P_1 in P. In the special case where $P = B$ is a block of KG we proved that already in 3.4.

A further consequence of the proof of Theorem C is

Proposition 4.3 *Let P be a projective indecomposable KG-module associated to $\varphi \in \operatorname{IBr}(G)$ and let Q be a lift of P to $\mathbb{C}G$. Then*

$$dim_K H^1(G, \Lambda^1(P)) = \begin{cases} \dim_{\mathbb{C}} S(Q)^G - \dim_{\mathbb{C}} \Lambda^2(Q)^G & \text{if } 1 \neq \varphi \\ \dim_{\mathbb{C}} S(Q)^G - \dim_{\mathbb{C}} \Lambda^2(Q)^G - 1 & \text{if } 1 = \varphi. \end{cases}$$

To decide whether a projective indecomposable module carries a non-degenerate G-invariant symplectic form is a highly difficult task. Some partial answers have been given in [1]. Here we show

Proposition 4.4 *Let P be a projective indecomposable module. If $\mathrm{H}^1(G, \Lambda^2(P)) = 0$ then P does not carry a non-degenerate G-invariant symplectic form.*

Proof: We may assume that $P \cong P^*$ since otherwise there is no non-degenerate G-invariant bilinear form on P. Since $\mathrm{H}^1(G, \Lambda^2(P)) = 0$ we have by (10) that

$$\Lambda^2(P)^G / \operatorname{Im}\sigma = \mathrm{H}^1(G, \Lambda^2(P)) = 0.$$

Thus $\Lambda^2(P)^G = \mathrm{Im}\,\sigma$. Furthermore

$$\mathrm{Im}\,\sigma \subseteq (P \otimes_K P)^G = \mathrm{End}_{KG}(P) = \Delta.$$

As P is indecomposable and K is a splitting field the endomorphism ring Δ is local with $\Delta/J(\Delta) = K$ where $J(\Delta)$ denotes the Jacobson radical of Δ. If $b \in \Lambda^2(P^*)^G = \mathrm{Im}\,\sigma$ then $b = c + \bar{c}$ for some bilinear form c of P. If we write $c = k \cdot id + j$ with $k \in K$ and $j \in J(\Delta)$ then $\bar{c} = k \cdot id + \bar{j}$. Thus $b \in J(\Delta)$ which shows that b is degenerated. $\square$

The condition $\mathrm{H}^1(G, \Lambda^2(P)) = 0$ in Proposition 4.4 can easily be tested using Theorem C.

Remark 4.5 The converse of Proposition 4.4 does not hold true in general. Let

$$H = \langle a, b \mid a^3 = b^4 = 1,\ a^b = a^{-1} \rangle$$

and put $G = HwrC_2$, the wreath product of H by C_2. Then G has a Brauer character φ of degree 4 such that P_φ does not carry a non-degenerate G-invariant symplectic form (see [1], 2.12). Thus, by Theorem C,

$$\dim_K \mathrm{H}^1(G, \Lambda^2(P_\varphi)) = \nu(\Phi_\varphi)$$

and, by ([7], 1.5), we have $\nu(\Phi_\varphi) = 2$.

Lemma 4.6 *Let Φ_φ denote again the classical projective character associated to $\varphi \in \mathrm{IBr}(G)$. Then*

$$\nu(\Phi_\varphi) \equiv c_{\varphi\overline{\varphi}} \mod 2.$$

Proof: If $P = P_\varphi$ then Theorem C says that

$$\dim_K \mathrm{H}^1(G, \Lambda^2(P)) = \nu(\Phi_\varphi) - \delta_{\varphi,1}.$$

According to (10) we have the exact sequence

$$0 \to \Lambda^2(P)^G \to (P \otimes_K P)^G \to (P^{(2)})^G \oplus \Lambda^2(P)^G \to \mathrm{H}^1(G, \Lambda^2(P)) \to 0, \tag{12}$$

and since $c_{\varphi\overline{\varphi}} = \dim_K (P \otimes_K P)^G$ we obtain

$$\dim_K \mathrm{H}^1(G, \Lambda^2(P)) \equiv c_{\varphi\overline{\varphi}} + \dim_K (P^{(2)})^G \mod 2.$$

If $P = P_1$ then $\dim_K (P^{(2)})^G = 1$, and if $P = P_\varphi$ with $\varphi \neq 1$ then $\dim_K (P^{(2)})^G = 0$. $\square$

Corollary 4.7 *If $2 \mid |G|$ then, for a block B, the dimension of $\mathrm{H}^1(G, \Lambda^2(B))$ is odd if and only if B is the principal block.*

Proof: We may assume that B is self-dual and by (1) we have

$$\dim \mathrm{H}^1(G, \Lambda^2(B)) = \sum_{\varphi \in \mathrm{IBr}(B)} \varphi(1) \dim \mathrm{H}^1(G, \Lambda^2(P_\varphi)).$$

If B is not the principal block then, by Lemma 4.6, we get

$$\begin{aligned} \dim \mathrm{H}^1(G, \Lambda^2(B)) &\equiv \textstyle\sum_{\varphi=\bar{\varphi}\in \mathrm{IBr}(B)} \varphi(1) c_{\varphi\varphi} + \sum_{\varphi \neq \bar{\varphi}} (\varphi(1) + \bar{\varphi}(1)) c_{\varphi\bar{\varphi}} \mod 2 \\ &\equiv \textstyle\sum_{\varphi=\bar{\varphi}\in \mathrm{IBr}(B)} \varphi(1) c_{\varphi\varphi} \mod 2 \\ &\equiv 0 \mod 2, \end{aligned}$$

since $2 \mid \varphi(1)$ for $1 \neq \varphi = \bar{\varphi} \in \mathrm{IBr}(G)$, by Fong's Lemma (see [3], Chap. VII, 8.13). For the principal block B the dimension of $\mathrm{H}^1(G, \Lambda^2(B))$ is odd since, by Lemma 4.6,

$$\dim \mathrm{H}^1(G, \Lambda^2(P_1)) = \nu(\Phi_1) - 1 \equiv c_{11} - 1 \mod 2$$

and c_{11} is always even if $2 \mid |G|$ (see [2], Proposition 3.6). □

Lemma 4.8 *For all $\varphi \in \mathrm{IBr}(G)$ we have*

a) $\nu(\Phi_\varphi) \leq c_{\varphi\overline{\varphi}}$.

b) Equality holds in a) if and only if $c_{\varphi,\overline{\varphi}} = \dim \Lambda^2(P_\varphi)^G + \delta_{\varphi,1}$.

c) If equality holds in a) for $\varphi \neq 1$ then all G-invariant forms on P_φ are symplectic. If in addition $\varphi = \overline{\varphi}$ then at least one such form is non-degenerated.

Proof: a) Let P be a projective indecomposable module. Since $\mathrm{H}^1(G, \Lambda^2(P))$ is a factor space of $\Lambda^2(P)^G$ and $\Lambda^2(P)^G \oplus (P^{(2)})^G \leq (P \otimes_K P)^G$ (as K-vector spaces) we get

$$\dim \mathrm{H}^1(G, \Lambda^2(P_\varphi)) \leq \dim \Lambda^2(P_\varphi)^G \leq c_{\varphi\overline{\varphi}}$$

for $\varphi \neq 1$, and

$$\dim \mathrm{H}^1(G, \Lambda^2(P_1)) \leq \dim \Lambda^2(P_1)^G \leq c_{11} - 1.$$

An application of Theorem C completes the proof.
b) Equality holds in a) if and only if

$$\Lambda^2(P_\varphi)^G \oplus (P_\varphi^{(2)})^G \cong (P_\varphi \otimes_K P_\varphi)^G \cong \mathrm{Hom}_{KG}(P_{\overline{\varphi}}, P_\varphi).$$

c) The first statement is clear from the proof of part a). The second follows since $\mathrm{End}_{KG}(P_\varphi)$ contains automorphisms. □

We call a KG-module V of *symplectic type* if V carries a G-invariant non-degenerate symplectic form.

Corollary 4.9

$$\sum_{\substack{\varphi \in \mathrm{IBr}(G) \\ P_\varphi \text{ of symplectic type}}} \varphi(1) \;\leq\; \sum_{\varphi \in \mathrm{IBr}(G)} \varphi(1)\nu(\Phi_\varphi) = 1 + |\,\mathrm{Inv}(G)| \;\leq\; \sum_{\varphi \in \mathrm{IBr}(G)} \varphi(1) c_{\varphi\overline{\varphi}}.$$

Proof: First note that $\mathrm{H}^1(G, \Lambda^2(P_\varphi)) = 0$ if φ does not belong to a self-dual block. Thus, by (1), we have

$$K\,\mathrm{Inv}(G) \cong \oplus_{\varphi\in\mathrm{IBr}(G)}\varphi(1)\,\mathrm{H}^1(G, \Lambda^2(P_\varphi))$$

To get the upper bound we apply Theorem C and Lemma 4.8. For the lower bound note that $\nu(\Phi_\varphi) \geq 0$ and, by Proposition 4.4, we have $\nu(\Phi_\varphi) > 0$ if P_φ is of symplectic type. $\square$

Examples 4.10 a) The group $G = \mathrm{S}_4$ has 9 involutions and the Cartan matrix of KG is $\begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$. Thus the upper bound in Corollary 4.9 is sharp.
b) For $G = C_3 \ltimes (C_2 \times C_2)$ the Cartan matrix is $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$, and therefore

$$|\,\mathrm{Inv}(G)| \leq 4 + 2 \cdot 2 - 1 = 7.$$

In this case the upper bound is sharp again. Clearly, G has two irreducible modules, the trivial module K and a 2-dimensional module, say V. Moreover, KG has two blocks, the principal block B_0 and a block B_1 containing V. One easily computes $\mathrm{H}^1(G, \Lambda^2(B_0)) \cong K \oplus K \oplus K$ and $\mathrm{H}^1(G, \Lambda^2(B_1) \cong V \oplus V$.
c) In contrast to $C_3 \ltimes (C_2 \times C_2)$ the group $C_3 \ltimes C_4$ has only one involution but the same Cartan matrix. Here we have

$$K\,\mathrm{Inv}(G) \cong \mathrm{H}^1(G, \Lambda^2(B_0)) \cong K.$$

Proposition 4.11 *If G has a normal Sylow 2-subgroup N then $|N| = \sum_{\varphi\in\mathrm{IBr}(G)} c_{\varphi\overline{\varphi}}$.*

Proof: Let H be a 2-complement in G (it exists by Zassenhaus' theorem). Since N is in the kernel of any irreducible KG-module we may identify $\mathrm{IBr}(G)$ with $\mathrm{Irr}(H)$. In particular, all irreducible Brauer characters have odd degree. Let β denote the character of $P_1|_H = (K_H)^G|_H$ and recall that for any $\varphi \in \mathrm{IBr}(G)$, the projective cover of the module V_φ affording φ satisfies $P_\varphi \cong V_\varphi \otimes P_1$. Thus

$$\begin{aligned} c_{\varphi\overline{\varphi}} &= \dim_K \mathrm{Hom}_{KG}(P_{\overline{\varphi}}, P_\varphi) = \dim \mathrm{Hom}_{KG}(V_{\overline{\varphi}} \otimes (K_H)^G, V_\varphi \otimes (K_H)^G) \\ &= \dim \mathrm{Hom}_{KG}(V_{\overline{\varphi}} \otimes (K_H)^G, (V_\varphi|_H)^G) = (\overline{\varphi}\beta, \varphi)_H = (\beta, \varphi^2)_H, \end{aligned}$$

and therefore

$$\sum_{\varphi\in\mathrm{IBr}(G)} c_{\varphi\overline{\varphi}} = \sum_{\varphi\in\mathrm{IBr}(G)} (\beta, \varphi^2)_H = (\beta, \sum_{\varphi\in\mathrm{IBr}(G)} \varphi^2)_H.$$

Furthermore, for any $1 \neq h \in H$, we have

$$\sum_{\varphi\in\mathrm{IBr}(G)} \varphi^2(h) = \sum_{\varphi\in\mathrm{IBr}(G)} \varphi(h)\overline{\varphi(h^{-1})} = 0$$

as h and h^{-1} can not be conjugated in a group of odd order. Since

$$\sum_{\varphi\in\mathrm{IBr}(G)} \varphi^2(1) = |H|$$

we obtain
$$\sum_{\varphi\in\mathrm{IBr}(G)} c_{\varphi\overline{\varphi}} = (\beta, \sum_{\varphi\in\mathrm{IBr}(G)} \varphi^2)_H = \beta(1) = \dim P_1 = |N|.$$

$\square$

Corollary 4.12 *If G has a normal elementary abelian Sylow 2-subgroup then*
$$1 + |\operatorname{Inv}(G)| = \sum_{\varphi\in\mathrm{IBr}(G)} c_{\varphi\overline{\varphi}}.$$

Surprisingly, the upper bound in Corollary 4.9 is achieved for some simple non-abelian groups with elementary abelian Sylow 2-subgroups.

Examples 4.13 a) For $G = \mathrm{A}_5$ the Cartan matrix is
$$\begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
There are 15 involutions in G. The 2-Brauer character degrees are $1, 2, 2, 4$. Thus
$$\sum_{\varphi\in\mathrm{IBr}(G)} \varphi(1)c_{\varphi\overline{\varphi}} \;=\; 4 + 2\cdot 2 + 2\cdot 2 + 4 = 16$$
and the upper bound is achieved in Corollary 4.9.
b) For $G = \mathrm{PSL}(2, 11)$ the Cartan matrix is
$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
with 2-Brauer character degrees $1, 5, 5, 10, 12, 12$. The 5-dimensional characters are conjugate to each other. Thus
$$\sum_{\varphi\in\mathrm{IBr}(G)} \varphi(1)c_{\varphi\overline{\varphi}} \;=\; 2 + 5 + 5 + 2\cdot 10 + 12 + 12 = 56.$$
Since G has exactly 55 involutions the upper bound in Corollary 4.9 is achieved again.
c) The smallest Janko group J_1 has 1463 involutions. Using the GAP library at Aachen we see that
$$\sum_{\varphi\in\mathrm{IBr}(J_1)} \varphi(1)c_{\varphi\overline{\varphi}} \;=\; 8+4\cdot 20+4\cdot 56+4\cdot 56+4\cdot 76+2\cdot 76+56+56+120+120+120 = 1464.$$
d) It is in general not true that in Corollary 4.9 equality holds for simple groups with elementary abelian Sylow 2-subgroups as the Ree group $R(27)$ shows. There are 512487 involutions in $R(27)$ whereas the upper bound in Corollary 4.9 yields 515296.

5 Metabolic G-spaces

Let V be a KG-module. The KG-module $\mathrm{M}(V) = V^* \oplus V$ endowed with a non-degenerate G-invariant symplectic form b which is totally zero on V^* is called a *metabolic G-space.* If $g \in G$ acts on V by the matrix $D(g)$ then g acts on V^* by $D(g^{-1})^t$ with respect to the dual basis (see [3], Chapter VII, 8.2).

Lemma 5.1 *Let $(\mathrm{M}(V), b)$ be a metabolic G-space. Given a K-basis of V we may choose a K-basis of V^* such that b has Gram matrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & B \end{pmatrix}$$

where B is the Gram matrix of the restriction $b|_V$ and $0, 1$ are the zero respectively the identity matrix of degree $\dim_K V$.

Proof: Let

$$\begin{pmatrix} 0 & A \\ A^t & B \end{pmatrix}$$

be the Gram matrix of b. Note that A is regular since b is non-degenerate. Since b is G-invariant we get $A \in \mathrm{End}_{KG}(V^*)$. Changing the basis of V^* via A^{-1} yields

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & B \end{pmatrix} \begin{pmatrix} (A^{-1})^t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & B \end{pmatrix}.$$

$\square$

According to Lemma 5.1 the metabolic G-spaces may be described by $(\mathrm{M}(V), b)$ where V is a KG-module and b is a symplectic G-invariant form on V. If b_1 and b_2 are G-invariant symplectic forms on V then we write $(V, b_1) \cong (V, b_2)$ resp. $(M(V), b_1) \cong (M(V), b_2)$ if (V, b_1) and (V, b_2) resp. $(M(V), b_1)$ and $(M(V), b_2)$ are isometric as G-modules.

Proposition 5.2 *Let $(M(V), b_1)$ and $(\mathrm{M}(V), b_2)$ be metabolic G-spaces.*

a) If $(V, b_2) \cong (V, b_1 + (b + \bar{b}))$ for some G-invariant bilinear form b of V then $(M(V), b_2) \cong (\mathrm{M}(V), b_1)$.

b) Suppose that V is an indecomposable KG-module. Then $(\mathrm{M}(V), b_2) \cong (\mathrm{M}(V), b_1)$ implies that $(V, b_2) \cong (V, b_1 + (b + \bar{b}))$ for some G-invariant bilinear form b of V.

Proof: a) Let B, B_1, B_2 be the Gram matrices for b, b_1 resp. b_2 with respect to a fixed basis of V. By assumption there exists a G-automorphism D of V such that

$$DB_2D^t = B_1 + (B + B^t).$$

Note that the matrix B defines a KG-linear map from V to V^*. Thus

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

is a KG-automorphism of $V^* \oplus V$. The equation

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}\begin{pmatrix} 1 & B^t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & B_1 + B + B^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & DB_2D^t \end{pmatrix}$$

shows that $(M(V), b_1) \cong (M(V), b_2')$ where b_2' denotes the bilinear form defined by DB_2D^t. On the other hand, since $(D^{-1})^t$ is a KG-automorphism of V^* and

$$\begin{pmatrix} (D^{-1})^t & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}\begin{pmatrix} D^{-1} & 0 \\ 0 & D^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & DB_2D^t \end{pmatrix}$$

we get that $(M(V), b_2) \cong (M(V), b_2')$ and the proof is complete.
b) Let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an isometry from $(\mathrm{M}(V), b_2)$ onto $(\mathrm{M}(V), b_1)$, hence

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}\begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}.$$

This implies the following equations.

(i) $BA^t + AB^t + BB_2B^t = 0$,

(ii) $BC^t + AD^t + BB_2D^t = 1$,

(iii) $DA^t + CB^t + DB_2B^t = 1$,

(iv) $DC^t + CD^t + DB_2D^t = B_1$.

If D is regular the last equation may be rewritten as

$$D[(D^{-1}C)^t + D^{-1}C + B_2]D^t = B_1.$$

Since $D^{-1}C$ is a KG-homomorphism from V^* to V it is the Gram matrix of a G-invariant bilinear form of V and we are done in this case since $D \in \mathrm{End}_{KG}(V)$ is invertible.

Thus we may assume that D is singular. We consider equation (ii), i.e.,

$$BC^t + AD^t + BB_2D^t = 1.$$

Observe that AD^t and BB_2D^t define maps in $\mathrm{End}_{KG}(V^*)$. Since $\mathrm{End}_{KG}(V^*)$ is a local algebra and D is singlar, the matrix B must be regular. Equation (i) now yields

$$BB_2B^t = AB^t + BA^t = AB^t + (AB^t)^t$$

or

$$B_2 = B^{-1}A + (B^{-1}A)^t.$$

Note that $B^{-1}A$ is a KG-homomorphism from V into V^*. Thus $b_2 = b + \bar{b}$ where b is the G-invariant form defined by the Gram matrix $B^{-1}A$. Interchanging b_1 and b_2 yields a G-invariant bilinear form c of V such that $b_1 = c + \bar{c}$. Thus we have proved that

$$b_2 = b_1 + (b + c) + (\overline{b + c})$$

for a G-invariant bilinear form $b + c$ of V. □

Proof of Theorem D. Let P be a projective indecomposable KG-module. According to (10), we have

$$\Lambda^2(P)^G / \operatorname{Im}\sigma \cong \mathrm{H}^1(G, \mathrm{S}^2(P)) = \mathrm{H}^1(G, \Lambda^2(P)).$$

By Proposition 5.2, the isometry classes of metabolic G-spaces $(M(P), b)$ are in one to one correspondence to the elements of $\Lambda^2(P^*)^G / \operatorname{Im}\sigma$ which completes the proof. □

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