ON HILBERT DIVISORS OF BRAUER CHARACTERS

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Abstract The Hilbert divisor $p^{a(\varphi)}$ of an irreducible p-Brauer character φ of a finite group G carries deep information about φ, respectively the module which affords φ. In [8] we conjectured that φ belongs to a p-block of defect 0 if and only if its Hilbert divisor is 1. In this note we continue our investigations.

Keywords: block, defect, Cartan matrix, Brauer character, Hilbert divisor, Green correspondent

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1. Introduction

Throughout the paper p is always a prime and G a finite group. We put $|G|_p = p^a$ and

$$G_{p'} = \{g \mid g \in G,\ g \text{ is a } p'\text{-element}\}.$$

By $\mathrm{IBr}_p(G)$ and $\mathrm{IBr}_p(B)$ we denote the set of irreducible p-Brauer characters of G, resp. of a p-block B of G with respect to a p-modular splitting system (K, R, k). Here R is a complete discrete valuation ring with unique maximal ideal πR, K is the quotient field of R of characteristic 0 and $k = R/\pi R$ the residue field of characteristic $p > 0$.

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Furthermore, let $R^* = R \setminus \pi R$ be the set of units in R. Similarly, we define $\mathrm{Irr}(G)$, resp. Irr(B) as the set of irreducible complex characters of G, resp. B. We write Φ_φ for the ordinary character associated to the projective cover of the module affording $\varphi \in \mathrm{IBr}_p(G)$. To be brief we call the ordinary character of a projective module a projective character. If χ is a generalized ordinary character of G, then χ° denotes the restriction of χ on $G_{p'}$. Let $vx(\varphi)$ denote the vertex of the module which affords φ. Finally, if $\varphi \in \mathrm{IBr}_p(B)$ where the block B has defect d, then the height $\mathrm{ht}(\varphi)$ of φ is defined by $\varphi(1)_p = p^{a-d+\mathrm{ht}(\varphi)}$. Note that in contrast to ordinary irreducible characters $\mathrm{ht}(\varphi) > d$ may happen.

In [8] we defined Hilbert divisors for irreducible Brauer characters, i.e., if $\varphi \in \mathrm{IBr}_p(G)$, then the Hilbert divisor of φ is the minimal p-power, say $p^{a(\varphi)}$, such that $p^{a(\varphi)}\varphi$ is a quasi-projective Brauer character which means that

$$p^{a(\varphi)}\varphi = \sum_{\psi \in \mathrm{IBr}_p(G)} a_\psi \Phi_\psi^\circ \text{ with } a_\varphi \in \mathbb{Z}. \tag{1}$$

If φ lies in the block B of defect d and $|vx(\varphi)| = p^v$, then the exponents $a(\varphi)$ of the Hilbert divisors satisfy $a(\varphi) \le v \le d$ and $a(\varphi) = d$ if $\mathrm{ht}(\varphi) = 0$ ([8], Theorem 2.1). According to Dickson's Theorem ([11], Corollary 2.14) we have $p^a \mid \Phi_\psi(1)$ for all $\psi \in \mathrm{IBr}_p(G)$. Thus from (1) we immediately get

$$\text{(i)} \quad d \le a(\varphi) + \mathrm{ht}(\varphi)$$
$$\text{(ii)} \quad p^{a-a(\varphi)} \mid \varphi(1).$$

Note that (ii) improves the well-known fact $p^{a-v} \mid \varphi(1)$. For instance, in the principal 2-block of the first simple Janko group J_1 there is an irreducible Brauer character φ of degree 56 with Hilbert divisor $2 = 2^{a(\varphi)}$, but $|vx(\varphi)| = 2^3$.

One of the main problems left open in [8] is the following.

Conjecture A. $\varphi \in \mathrm{IBr}_p(G)$ belongs to a block of defect 0 if and only if its Hilbert divisor is 1, i.e., $a(\varphi) = 0$.

The conjecture holds true in the following cases:

a) for p-solvable groups ([8], Proposition 2.7),
b) for p-blocks with cyclic defect groups ([8], Proposition 2.9),
c) for blocks of tame representation type according to Erdmann's work [2].

2. Brauer characters of height zero

Throughout this section let B be a p-block of G of defect d and let $\varphi \in \mathrm{IBr}_p(B)$.

Lemma 2.1. *If* $\mathrm{ht}(\varphi) = 0$*, then* $a(\varphi) = d$*.*

Proof. This is ([8],Theorem 2.1 c)). □

Lemma 2.2. *If* G *is* p*-solvable, then* $a(\varphi) + \mathrm{ht}(\varphi) = d$*.*

Proof. The following equation

$$\begin{aligned} \Phi_\varphi(1) &= p^a \varphi(1)_{p'} && \text{(by ([3], Chap. X, Theorem 3.2))} \\ &= p^{a(\varphi)}\varphi(1) && \text{(by ([8], Proposition 2.8))} \\ &= p^{a(\varphi)+a-d+\mathrm{ht}(\varphi)}\varphi(1)_{p'} \end{aligned}$$

implies $a(\varphi) + \mathrm{ht}(\varphi) = d$. □

We would like to mention here that the conclusion in Lemma 2.2 very often, but not always holds true for groups which are not p-solvable.

Corollary 2.3. *For a* p*-solvable group we have* $a(\varphi) = d$ *if and only if* $\mathrm{ht}(\varphi) = 0$*.*

Example 2.4. Let G be the alternating group A_9 of degree 9 and let $B = B_0$ be the principal 3-block of G. According to [9] the block B contains five irreducible Brauer characters of degree $1, 7, 21, 35, 41$. Note that only the third character has non-zero height. For $a(\varphi)$ one easily computes $4, 4, 3, 4, 4$.

Question 2.5. *Does* Corollary 2.3 *hold true without the assumption of* p*-solvability?*

Lemma 2.6. *If* $\varphi \in \mathrm{IBr}_p(B)$*, then the following are equivalent.*

a) $p^{a(\varphi)}\varphi$ *is of height* 0*.*
b) B *is a block of defect* 0*.*

Proof. If $p^{a(\varphi)}\varphi$ is of height 0, then $a - d = a(\varphi) + a - d + \mathrm{ht}(\varphi)$, hence $a(\varphi) = 0 = \mathrm{ht}(\varphi)$ and b) follows directly from ([8], Proposition 2.10). The other direction is clear. □

3. Characterization of Hilbert divisors

Theorem 3.1. *If* $\varphi \in \mathrm{IBr}_p(G)$*, then*

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|_p} \in R$$

for all $x \in G_{p'}$ *and there exists an* x *such that* $\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|_p} \in R^*$*.*

Proof. Note that

$$p^{a(\varphi)}\varphi(x) = \sum_{\psi \in \mathrm{IBr}_p(B)} a_\psi \Phi^\circ_\psi(x) \quad (a_\psi \in \mathbb{Z}, x \in G_{p'}),$$

by the definition of Hilbert divisors. Thus

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|_p} = \sum_{\psi \in \mathrm{IBr}_p(B)} a_\psi \frac{\Phi^\circ_\psi(x)}{|C_G(x)|_p}.$$

Now ([11], Lemma 2.21) says that

$$\frac{\Phi_\psi(x)}{|C_G(x)|_p} \in R$$

for $x \in G_{p'}$ and the first part of the assertion has been proved. To get the second part we have to prove that

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|_p} \notin \pi R$$

for some $x \in G_{p'}$. Suppose that $\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|_p} \in \pi R$ for all $x \in G_{p'}$. Thus, if $\langle \cdot\,, \cdot \rangle^\circ$ denotes the usual scalar product of the K-class functions on $G_{p'}$, then

$$\langle p^{a(\varphi)}\varphi, \psi\rangle^\circ = \frac{1}{|G|} \sum_{x \in G_{p'}} p^{a(\varphi)}\varphi(x)\psi(x^{-1}) = \sum_{i=1}^{h} \frac{p^{a(\varphi)}\varphi(x_i)}{|C_G(x_i)|}\psi(x_i^{-1}) \in \pi R$$

for all $\psi \in \mathrm{IBr}_p(G)$, a contradiction to the following lemma. □

Lemma 3.2. *Let $\varphi \in \mathrm{IBr}_p(B)$. Then there exists a $\psi \in \mathrm{IBr}_p(B)$ such that*

$$\langle p^{a(\varphi)}\varphi, \psi\rangle^\circ \in R^*.$$

Proof. Let $C^{-1} = (c^{\varphi\psi})$ denote the inverse of the Cartan matrix of B. Then

$$\langle p^{a(\varphi)}\varphi, \psi\rangle^\circ = p^{a(\varphi)}c^{\varphi\psi}.$$

By the construction of the Hilbert divisors (see part a) of the proof of ([8], Theorem 2.1)) we see that $p^{a(\varphi)}c^{\varphi\psi} \in \mathbb{Z}$ for all $\psi \in \mathrm{IBr}_p(B)$ and there exists a $\psi \in \mathrm{IBr}_p(B)$ such that $p \nmid p^{a(\varphi)}c^{\varphi\psi}$. □

Let $\hat{}: R \longrightarrow R/\pi R = k$ denote the natural epimorphism. Note that for $\varphi \in \mathrm{IBr}_p(G)$ and $x \in G_{p'}$ the value $\widehat{\varphi(x)}$ is the trace of x with respect to the k-representation affording φ.

Corollary 3.3. *If $\varphi \in \mathrm{IBr}_p(G)$ and $\widehat{\varphi(x)} \neq 0$ for $x \in G_{p'}$, then* $|C_G(x)|_p \mid p^{a(\varphi)}$.

Proof. $\widehat{\varphi(x)} \neq 0$ means that $\varphi(x) \in R^*$. Thus Theorem 3.1 implies the assertion. □

Example 3.4. In general there is no $x \in G_{p'}$ with $|C_G(x)|_p = p^{a(\varphi)}$. As an example we may take the alternating group $G = A_5$ and $p = 2$. In this case there exists $\varphi \in \mathrm{IBr}_2(G)$ with $\varphi(1) = 2$ and $a(\varphi) = 1$. On the other hand $|C_G(x)|_2 = 1$ for all $1 \neq x \in G_{2'}$.

Corollary 3.5. *If $\varphi \in \mathrm{IBr}_p(G)$ with $a(\varphi) = 0$, then there exists an $x \in G_{p'}$ of p-defect 0 such that $\varphi(x) \in R^*$.*

Theorem 3.1 immediately leads to the following characterization of Hilbert divisors.

Theorem 3.6. *If $\varphi \in \mathrm{IBr}_p(G)$, then $a(\varphi)$ is the smallest non-negative integer n such that $\frac{p^n\varphi(x)}{|C_G(x)|_p} \in R$ for all $x \in G_{p'}$ and there exists an $x \in G_{p'}$ such that $\frac{p^n\varphi(x)}{|C_G(x)|_p} \in R^*$.*

With the Theorem above we can reformulate Conjecture A to a statement which obviously extends a classical result, namely that $\chi \in \mathrm{Irr}(G)$ belongs to a block of defect 0 if $\frac{\chi(1)}{|C_G(1)|_p} \in R^*$.

Conjecture A*. If $\varphi \in \mathrm{IBr}_p(G)$, then the following are equivalent.

a) φ lies in a p-block of defect 0.
b) $\frac{\varphi(x)}{|C_G(x)|_p} \in R$ for all $x \in G_{p'}$ and for at least one x we have $\frac{\varphi(x)}{|C_G(x)|_p} \in R^*$.

Question 3.7. *Do we always have*

$$\varphi(1)_p \leq p^{a+a(\varphi)}$$

for $\varphi \in \mathrm{IBr}_p(G)$ where $p^a = |G|_p$?

Remark 3.8. If the bound in Question 3.7 holds true, then it is sharp. The McLaughlin group $G = McL$ has an irreducible Brauer character φ in the principal 2-block with $\varphi(1)_2 = 2^9$ (see [9]). Note that $|G|_2 = 2^7$. One easily computes $a(\varphi) = 2$ which shows that the bound is sharp.

4. Defect of irreducible Brauer characters

Let B be a p-block of G of defect d. Then the defect $d(\chi)$ of $\chi \in \mathrm{Irr}(B)$ is defined by $d(\chi) = d - \mathrm{ht}(\chi)$. If we replace χ by a Brauer character φ, this number may be negative. Thus, for Brauer characters we define the defect as follows.

Definition 4.1. If $\varphi \in \mathrm{IBr}_p(B)$, then we call

$$d(\varphi) = a(\varphi) + \mathrm{ht}(\varphi) - d$$

the defect of φ.

Note that by (i) in the introduction, $d(\varphi) \geq 0$ and $d(\varphi) = 0$ if and only if $d = a(\varphi) + \mathrm{ht}(\varphi)$. If G is p-solvable or B has a cyclic defect group, then $d(\varphi) = 0$ by Lemma 2.2, resp. ([8], Proposition 2.9) together with [13].

For $x \in G$, we denote by $\tilde{x}$ the sum of all elements conjugate to x in G. Recall that for $\chi \in \mathrm{Irr}(G)$, we have

$$\omega_\chi(\tilde{x}) = \frac{|G : C_G(x)|\chi(x)}{\chi(1)} \in R$$

for all $x \in G$ (see for instance ([10], Chap. III, Theorem 2.25)). If we replace χ by $\varphi \in \mathrm{IBr}_p(G)$, then in general

$$\omega_\varphi(\tilde{x}) = \frac{|G : C_G(x)|\varphi(x)}{\varphi(1)}$$

need not to be in R for $x \in G_{p'}$ as [15] shows. Instead we have

Theorem 4.2. *If $\varphi \in \mathrm{IBr}_p(B)$ where B is a p-block, then*

$$p^{d(\varphi)}\omega_\varphi(\tilde{x}) \in R$$

for all $x \in G_{p'}$ and $p^{d(\varphi)}\omega_\varphi(\tilde{x}) \in R^*$ for at least one $x \in G_{p'}$.

Proof. We have

$$\begin{aligned} p^{d(\varphi)}\omega_\varphi(\tilde{x}) &= \frac{p^{d(\varphi)}|G:C_G(x)|\varphi(x)}{\varphi(1)} \\ &= \frac{p^{a(\varphi)+\mathrm{ht}(\varphi)-d}|G|\varphi(x)}{p^{a-d+\mathrm{ht}(\varphi)}\varphi(1)_{p'}|C_G(x)|} \\ &= \frac{|G|_{p'}}{\varphi(1)_{p'}} \cdot \frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|}. \end{aligned}$$

Thus the assertion follows by Theorem 3.1.

□

As a consequence of Theorem 4.2 we get an early result of Okuyama.

Lemma 4.3. ([12], Lemma 2.1) *If $\varphi \in \mathrm{IBr}_p(G)$ is of height 0, then $\omega_\varphi(\tilde{x}) \in R$ for all $x \in G_{p'}$.*

Proof. The condition $\mathrm{ht}(\varphi) = 0$ implies $a(\varphi) = d$ (see ([8], Theorem 2.1)) where d is the defect of the block to which φ belongs. Thus $d(\varphi) = a(\varphi) + \mathrm{ht}(\varphi) - d = 0$ and Theorem 4.2 leads to the assertion. □

Remark 4.4. a) In general $p^{d(\varphi)}\omega_\varphi(\tilde{x}) \not\equiv p^{d(\psi)}\omega_\psi(\tilde{x}) \bmod \pi R$ if φ and ψ are irreducible Brauer characters in the same p-block: The principal 2-block of $G = J_1$ has an irreducible Brauer character φ of degree $56 = 7 \cdot 2^3$ with $a(\varphi) = 1$ and $\mathrm{ht}(\varphi) = 3$, hence $d(\varphi) = 1$. But the trivial Brauer character 1_G has defect $d(1_G) = 0$. Thus $p^{d(\varphi)}\omega_\varphi(\tilde{1}) = 2$ and $p^{d(1_G)}\omega_{1_G}(\tilde{1}) = 1$.
b) Suppose that $d(\varphi) = 0$ for $\varphi \in \mathrm{IBr}_p(G)$. Thus

$$\lambda_\varphi(\tilde{x}) = \omega_\varphi(\tilde{x}) + \pi R = k$$

for all $x \in G_{p'}$. In general λ_φ does not define the central character of kG with respect to the p-block to which φ belongs. For instance, one may take $G = A_{14}$ and $p = 2$. According to GAP [4] the principal 2-block of G contains 15 irreducible Brauer characters including φ_1 and φ_{21} (in notation of GAP these are $X1$ and $X21$) with

$$(a(\varphi_i), \mathrm{ht}(\varphi_i)) = (10, 0),\ (3, 7)$$

for $i = 1, 21$, hence $d(\varphi_i) = 0$. However the functions λ_{φ_i} are different. This answers Feit's question (I) in ([3], Chap. IV, section 5, page 166) in the negative.

5. Normal subgroups

By ([3], Chap. III, Corollary 4.13) we know that a normal p-subgroup N of G is contained in the vertex of any irreducible Brauer character φ of G. Moreover $|vx(\varphi)| = |N||vx(\bar{\varphi})|$ where $\bar{\varphi}$ is the Brauer character of the module affording φ but regarded as a module of $\bar{G} = G/N$. Note that N is contained in the kernel of φ.

Proposition 5.1. *Let N be a normal subgroup of G with $|N|_p = p^n$, $|G|_p = p^a$ and $\bar{G} = G/N$. Suppose that $\varphi \in \mathrm{IBr}_p(G)$. Then we have the following.*

a) $a(\varphi) \le a(\bar{\varphi}) + n$.
b) $a(\bar{\varphi}) + n - \mathrm{ht}(\bar{\varphi}) \le a(\varphi)$.
c) *If $d(\bar{\varphi}) = 0$, then $a(\varphi) = a(\bar{\varphi}) + n$.*

Proof. a) We prove that

$$\check{\varphi}(g) = \begin{cases} p^{a(\overline{\varphi})+n}\varphi(g), & \text{if } g \text{ is a } p\text{'-element,} \\ 0, & \text{otherwise} \end{cases}$$

is a generalized character of G, so that $a(\varphi) \le a(\overline{\varphi}) + n$ because of the minimality of $a(\varphi)$. According to Brauer's characterization of generalized characters ([3], Chap. IV, Theorem 1.1), it suffices to prove that $\check{\varphi}_E$ is a generalized character for any elementary subgroup E of G. Let

$E = P \times H$, where P and H are p- and p'-subgroups of G respectively. Let $\xi \in \mathrm{Irr}(P)$ and $\eta \in \mathrm{Irr}(H)$. We have

$$\begin{aligned}(\check{\varphi}_E, \xi \times \eta)_E &= \frac{1}{|E|} \sum_{y \in H} \sum_{x \in P} \check{\varphi}(xy) \overline{\xi(x)\eta(y)} \\ &= \xi(1) \cdot \frac{p^{a(\overline{\varphi})+n}}{|E|} \sum_{y \in H} \varphi(y)\overline{\eta(y)}.\end{aligned}$$

However, we know that the inflation $\mathrm{Inf}(\widetilde{\overline{\varphi}})$ of

$$\widetilde{\overline{\varphi}}(g) = \begin{cases} p^{a(\overline{\varphi})}\overline{\varphi}(\overline{g}), & \text{if } \overline{g} \text{ is a } p\text{'-element of } \overline{G}, \\ 0, & \text{otherwise} \end{cases}$$

is a generalized character of G. Computing $((\mathrm{Inf}(\widetilde{\overline{\varphi}}))_E, 1_P \times \eta)_E$, we have

$$\begin{aligned}((\mathrm{Inf}(\widetilde{\overline{\varphi}}))_E, 1_P \times \eta)_E &= \frac{1}{|E|} \sum_{x \in P, y \in H} (\mathrm{Inf}(\widetilde{\overline{\varphi}}))(xy)\overline{1_P(x)\eta(y)} \\ &= \frac{1}{|E|} \sum_{x \in P \cap N, y \in H} (\mathrm{Inf}(\widetilde{\overline{\varphi}}))(xy)\overline{\eta(y)} \\ &= \frac{p^{a(\overline{\varphi})}|P \cap N|}{|E|} \sum_{y \in H} \varphi(y)\overline{\eta(y)},\end{aligned}$$

and so $\frac{p^{a(\overline{\varphi})+n}}{|E|} \sum_{y \in H} \varphi(y)\overline{\eta(y)}$ is an integer. Thus $\check{\varphi}_E$ is a generalized character of E, as desired.

b), c) Let B be the block of defect d to which φ belongs and let $\bar{B}$ be the block of defect $\bar{d}$ to which $\bar{\varphi}$ belongs. Note that

$$p^{a-d+\mathrm{ht}(\varphi)} = \varphi(1)_p = \bar{\varphi}(1)_p = p^{\bar{a}-\bar{d}+\mathrm{ht}(\bar{\varphi})}. \tag{2}$$

By ([11], Theorem 9.9 (a)), we have

$$d - \bar{d} = n + m \tag{3}$$

for some $m \geq 0$. Thus equation (2) implies that

$$\mathrm{ht}(\varphi) = \mathrm{ht}(\bar{\varphi}) + m. \tag{4}$$

Since $p^{a(\varphi)}\varphi$ is quasi-projective we get

$$p^a \mid p^{a(\varphi)}\varphi(1)_p = p^{a(\varphi)+a-d+\mathrm{ht}(\varphi)},$$

hence

$$a(\varphi) \geq d - \mathrm{ht}(\varphi).$$

By (3) and (4), it follows

$$a(\varphi) \geq \bar{d} + n + m - (\mathrm{ht}(\bar{\varphi}) + m) \geq a(\bar{\varphi}) + n - \mathrm{ht}(\bar{\varphi})$$

and in case $d(\bar{\varphi}) = 0$, i.e., $a(\bar{\varphi}) + \mathrm{ht}(\bar{\varphi}) = \bar{d}$, we obtain

$$a(\varphi) \geq \bar{d} + n + m - (\mathrm{ht}(\bar{\varphi}) + m) = a(\bar{\varphi}) + n.$$

□

Note that the left hand side of the inequality of Proposition 5.1 b) might be negative.

Theorem 5.2. *Let N be a normal p-subgroup of G of order p^n, let $\bar{G} = G/N$ and let $\varphi \in \mathrm{IBr}_p(G)$. Then the following are equivalent.*

a) $a(\varphi) = a(\bar{\varphi}) + n$.

b) *For $x \in G_{p'}$ we have*

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|} \in R^* \Longleftrightarrow x \in C_G(N) \text{ and } \frac{p^{a(\bar{\varphi})}\bar{\varphi}(\bar{x})}{|C_{\bar{G}}(\bar{x})|} \in R^*.$$

Proof. a) $\Longrightarrow$ b) Let $x \in G_{p'}$ such that $\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|} \in R^*$. Thus, if ν denotes the valuation on K with $\nu(p) = 1$, then

$$a(\varphi) + \nu(\varphi(x)) = \nu(|C_G(x)|).$$

Since

$$C_G(x)/C_N(x) \cong C_G(x)N/N \leq C_{\bar{G}}(\bar{x})$$

we get

$$|C_G(x)| \mid |C_{\bar{G}}(\bar{x})||C_N(x)|.$$

It follows that

$$\begin{aligned} a(\varphi) + \nu(\varphi(x)) &= \nu(|C_G(x)|) \\ &\overset{(i)}{\leq} \nu(|C_{\bar{G}}(\bar{x})|) + \nu(|C_N(x)|) \\ &\overset{(ii)}{\leq} a(\bar{\varphi}) + \nu(\bar{\varphi}(\bar{x})) + n. \end{aligned}$$

Since $\nu(\varphi(x)) = \nu(\bar{\varphi}(\bar{x}))$ and by assumption $a(\varphi) = a(\bar{\varphi}) + n$, we have equality in (i) and (ii). Hence

$$(i') \qquad C_N(x) = N$$

and

$$(ii') \qquad a(\bar{\varphi}) + \nu(\bar{\varphi}(\bar{x})) = \nu(|C_{\bar{G}}(\bar{x})|).$$

Note that (ii') means that $\frac{p^{a(\bar{\varphi})}\bar{\varphi}(\bar{x})}{|C_{\bar{G}}(\bar{x})|} \in R^*$.

Vice versa, if (i') and (ii') holds then

$$\begin{aligned} a(\varphi) + \nu(\varphi(x)) &= a(\bar{\varphi}) + \nu(\bar{\varphi}(\bar{x})) + n \\ &= \nu(|C_{\bar{G}}(\bar{x})|) + \nu(|C_N(x)|) \\ &= \nu(|C_G(x)|) \\ &\leq a(\varphi) + \nu(\varphi(x)), \end{aligned}$$

hence

$$a(\varphi) + \nu(\varphi(x)) = \nu(|C_G(x)|)$$

which shows that

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|} \in R^*.$$

b) $\Longrightarrow$ a) We have

$$\begin{aligned} a(\varphi) + \nu(\varphi(x)) &= \nu(|C_G(x)|) \\ &= \nu(|C_{\bar{G}}(\bar{x})|) + \nu(|N|) \\ &= a(\bar{\varphi}) + \nu(\bar{\varphi}(\bar{x})) + n. \end{aligned}$$

Thus $a(\varphi) = a(\bar{\varphi}) + n$. □

Proposition 5.3. *Let N be a normal p-subgroup of G and let $\bar{G} = G/N$. If $\varphi \in \mathrm{IBr}_p(G)$, then $a(\bar{\varphi}) \leq a(\varphi)$.*

Proof. By induction we may assume that N is abelian. Due to the Alperin-Collins-Sibley Theorem ([7], Chap. II, Theorem 11.14) we have

$$\Phi_\varphi(x) = \rho(x)\Phi_{\bar{\varphi}}(\bar{x})$$

for all $x \in G_{p'}$ where ρ is the Brauer character afforded by the conjugation action of G on kN. Since N acts trivially on kN we obtain $\rho(x) = \rho(\bar{x})$. Thus we have

$$p^{a(\varphi)}\varphi(x) = \sum_{\psi \in \mathrm{IBr}_p(G)} a_\psi \Phi_\psi(x) = \sum_{\bar{\psi} \in \mathrm{IBr}_p(\bar{G})} a_{\bar{\psi}} \rho(\bar{x}) \Phi_{\bar{\psi}}(\bar{x}).$$

Note that $\rho \cdot \Phi_{\bar{\psi}}$ is a sum of projective characters of $\bar{G}$ ([11], Lemma 2.25), hence

$$\rho(\bar{x})\Phi_{\bar{\psi}}(\bar{x}) = \sum_{\bar{\lambda} \in \mathrm{IBr}_p(\bar{G})} b_{\bar{\lambda}} \Phi_{\bar{\lambda}}(\bar{x}).$$

Therefore we obtain

$$p^{a(\varphi)}\varphi(x) = \sum_{\bar{\psi} \in \mathrm{IBr}_p(\bar{G})} \sum_{\bar{\lambda} \in \mathrm{IBr}_p(\bar{G})} a_{\bar{\psi}} b_{\bar{\lambda}} \Phi_{\bar{\lambda}}(\bar{x}).$$

Since N is in the kernel of φ we finally have

$$p^{a(\varphi)}\bar{\varphi}(\bar{x}) = p^{a(\varphi)}\varphi(x) = \sum_{\bar{\psi} \in \mathrm{IBr}_p(\bar{G})} \sum_{\bar{\lambda} \in \mathrm{IBr}_p(\bar{G})} a_{\bar{\psi}} b_{\bar{\lambda}} \Phi_{\bar{\lambda}}(\bar{x})$$

with $a_{\bar{\psi}} b_{\bar{\lambda}} \in \mathbb{Z}$. This implies $a(\bar{\varphi}) \leq a(\varphi)$.

□

Example 5.4. Let $G = 3^6 : 2M_{12}$ where M_{12} is the Mathieu group on 12 letters and let $p = 3$. Then G consists only of the principal 3-block and for $a(\varphi)$ where $\varphi \in \mathrm{IBr}_3(G)$ one computes

$$9, 9, 9, 9, 8, 8, 7, 7, 9, 9, 9, 9, 8, 8, 8, 8, 7, 7, 6.$$

The factor group $\bar{G} = 2M_{12}$ has exactly four blocks and we compute for $a(\bar{\varphi})$

block 1:	3,3,3,3,2,2,1,1	block 2:	3,3,3,3,2,2,2,2
block 3:	1,1	block 4:	0

Thus we get $a(\varphi) = a(\bar{\varphi}) + 6$ for all φ.

Many other examples lead to the following conjecture.

Conjecture 5.5. If N is a normal p-subgroup of G of order p^n, then $a(\varphi) = a(\bar{\varphi}) + n$ for all $\varphi \in \mathrm{IBr}_p(G)$.

The conjecture holds true for p-solvable groups, by Lemma 2.2 and Proposition 5.1 part c).

6. The Green correspondence

Let $\mathbb{Z}[\mathrm{IBr}_p(G)] = \{\sum_{\varphi \in \mathrm{IBr}_p(G)} a_\varphi \varphi \mid a_\varphi \in \mathbb{Z}\}$ denote the ring of $\mathbb{Z}$-linear combinations of irreducible Brauer characters. As for irreducible Brauer characters we may define Hilbert divisors $p^{a(\psi)}$ for any $\psi \in \mathbb{Z}[\mathrm{IBr}_p(G)]$ with $a(\psi) \leq d$ in case $\psi \in \mathbb{Z}[\mathrm{IBr}_p(B)]$ where B is a p-block of defect d (see the proof of Theorem 2.1 in [8]).

Lemma 6.1. *Let $H \leq G$.*

a) *If $\psi = \sum_{\varphi \in \mathrm{IBr}_p(G)} a_\varphi \varphi \in \mathbb{Z}[\mathrm{IBr}_p(G)]$, then $a(\psi) \leq \max\{a(\varphi) \mid a_\varphi \neq 0\}$ and $a(\psi|_H) \leq a(\psi)$.*
b) *If $\lambda \in \mathbb{Z}[\mathrm{IBr}_p(H)]$, then $a(\lambda^G) \leq a(\lambda)$.*

Proof. This is a consequence of the fact that induction and restriction of projective modules are projective. □

Proposition 6.2. *Let M be an indecomposable kG-module with Brauer character φ. Then $p^{a(\varphi)} \leq |vx(\varphi)|$.*

Proof. The proof is exactly the same as for simple modules (see Proposition 2.6 in [8]). □

Theorem 6.3. *Let M be an indecomposable kG-module with Brauer character φ and $|vx(\varphi)| = p^v$. Let $f(\varphi)$ denote the Brauer character of the Green correspondent of M in $(G, V, H = N_G(V))$. Then $a(\varphi) = v$ if and only if $a(f(\varphi)) = v$.*

Proof. We may assume that $v > 1$. First we consider the case that $a(\varphi) = v$. According to the Green correspondence ([7], Chap. II, Theorem 4.1) or ([10], Chap. 4, Theorem 4.3 b)) we have

$$f(M)^G = M \oplus o(\mathcal{X})$$

where the indecomposable direct summands in $o(\mathcal{X})$ have vertices in

$$\mathcal{X} = \{W \leq G \mid W \leq V \cap V^g,\ g \in \mathrm{G} \setminus H\}.$$

Thus for Brauer characters we get the equation

$$\beta^G = \varphi + \psi$$

where β and ψ are the Brauer characters of $f(M)$ and $o(\mathcal{X})$ respectively. Multiplying this equation by p^{v-1} we get

$$(p^{v-1}\beta)^G = p^{v-1}\varphi + p^{v-1}\psi.$$

Suppose that $a(\beta) \leq v - 1$. Then $(p^{v-1}\beta)^G$ is quasi-projective. Since all vertices occurring in the direct summands of $o(\mathcal{X})$ have order less than or equal to $v - 1$ we get, by Proposition 6.2, that $p^{v-1}\psi$ is also quasi-projective. Thus $p^{v-1}\varphi$ is quasi-projective, a contradiction.

Now let $a(f(\varphi)) = a(\beta) = v$. By ([10], Chap. 4, Lemma 4.2 (i)) we have

$$(f(M)^G)|_H = f(M) \oplus o(\mathcal{Y})$$

where the indecomposable direct summands in $o(\mathcal{Y})$ have vertices in

$$\mathcal{Y} = \{W \leq G \mid W \leq V^g \cap H,\ g \in \mathrm{G} \setminus H\}.$$

Thus in terms of Brauer characters we

$$\beta^G|_H = \beta + \eta$$

where η is the the Brauer character of $o(\mathcal{Y})$. Applying ([10], Chap. 4, Lemma 4.2 (ii) (a)) we see that all indecomposable components of $\mathcal{Y}$ have a vertex of order less than p^v. Now we assume that $a(M) = a(\varphi) \leq v - 1$. Since $\beta^G = \varphi + \psi$ we get

$$a(\beta^G|_H) \leq a(\beta^G) \leq \max\{a(\varphi), a(\psi)\} \leq v - 1$$

and the equation $\beta^G|_H = \beta + \eta$ leads to $a(f(M)) = a(\beta) \leq v - 1$, a contradiction. □

Corollary 6.4. *Let G be a p-solvable group and let $f(\varphi)$ denote the Green correspondent of $\varphi \in \mathrm{IBr}_p(G)$ in $(G, vx(\varphi), N_G(vx(\varphi))$. Then $a(\varphi) = a(f(\varphi))$.*

Proof. By ([8], Proposition 2.7), we have $p^{a(\varphi)} = |vx(\varphi)|$ and we may apply Theorem 6.3. □

Example 6.5. a) Let $G = A_5$ be the alternating group on 5 letters and $p = 2$. Then one easily computes $a(\varphi) = a(f(\varphi))$ for all $\varphi \in \mathrm{IBr}_2(G)$ where $f(\varphi)$ is the Brauer character of the Green correspondent of the module affording φ in $(G, vx(\varphi), N_G(vx(\varphi))$.
b) Let $G = M_{12}$ be the Mathieu group on 12 letters and let $p = 2$. We have $|G|_2 = 2^6$. Again $a(\varphi) = a(f(\varphi)) = 6, 5, 4$ for $\varphi \in \mathrm{IBr}_2(B_0)$ of dimension $1, 10, 44$ in the principal 2-block B_0. For the other block B_2 with $d = 2$ we have $a(\varphi) = a(f(\varphi)) = 2$ for all $\varphi \in \mathrm{IBr}_2(B_2)$. The computations are based on the information given in [14].
c) The group $G = M_{22}$ has only one 2-block. The degrees of the irreducible Brauer characters are $1, 10, 10, 34, 70, 70, 98$ and all vertices are equal to the Sylow 2-subgroup of G which is of order 2^7. For $a(\varphi)$ one computes $7, 6, 6, 6, 6, 6, 6$ and for $a(f(\varphi))$ as well $7, 6, 6, 6, 6, 6, 6$. The computations are again based on the information given in [14].
d) Let $G = M_{24}$ and $p = 3$. The principal 3-block B_0 has irreducible Brauer characters of degrees $1, 22, 231, 483, 770, 770, 1243$. Apart from the character of degree 483 which has a vertex of order 9, all have a Sylow 3-subgroup (of order 27) as vertex. The exponents $a(\varphi)$ are $3, 3, 2, 2, 3, 3, 3$. With the information of [1] we again have $a(\varphi) = a(f(\varphi))$ for all $\varphi \in \mathrm{IBr}_3(B_0)$. The information about the Green correspondents may be taken from [1].

7. p-solvable groups

It is well-known that the Cartan numbers of a p-block B of defect d of a p-solvable group are bounded by p^d (see for instance ([3], Chap. X, Theorem 4.2) or [6]). Actually a stronger result holds true if B contains at least two irreducible Brauer characters.

Proposition 7.1. *Let B be a p-block of the p-solvable group of defect d and let $C = (c_{\varphi\psi})$ denote the Cartan matrix of B. If $l(B) \geq 2$, then $c_{\varphi\psi} < p^d$ for all $\varphi, \psi \in \mathrm{IBr}_p(B)$.*

Proof. Clearly, $d > 0$ since B contains at least two irreducible Brauer characters. Suppose that $c_{\varphi\psi} = p^d$ for some $\varphi, \psi \in \mathrm{IBr}_p(B)$. As

$$p^d = c_{\varphi\psi} \leq \max\{|vx(\varphi)|, |vx(\psi)|\} \leq p^d$$

by [6], we may assume that $p^d = |vx(\varphi)| \geq |vx(\psi)|$. According to [3, Chap. X, Theorem 1.8] we also have

$$\Phi_\beta(1) = |vx(\beta)|\beta(1)$$

for all $\beta \in \mathrm{IBr}_p(B)$. Since

$$\Phi_\varphi(1) = c_{\varphi\varphi}\varphi(1) + c_{\varphi\psi}\psi(1) + \ldots \;\geq\; p^d\psi(1) \;\geq\; |vx(\psi)|\psi(1) = \Phi_\psi(1) \tag{5}$$

we get

$$\Phi_\varphi(1) \geq \Phi_\psi(1).$$

On the other hand

$$\Phi_\psi(1) = c_{\psi\psi}\psi(1) + c_{\varphi\psi}\varphi(1) + \dots \ \geq \ p^d\varphi(1) = \Phi_\varphi(1)$$

leads to $\Phi_\psi = \Phi_\varphi$ and hence $\varphi(1) \geq \psi(1)$ since $p^d = |vx(\varphi)| \geq |vx(\psi)|$. From this we see that in equation (5) everywhere holds equality which implies $c_{\varphi\varphi} = 0$, a contradiction. □

Proposition 7.2. *Let G be a p-solvable group and let B be a p-block of defect d. If the defect group of B is abelian, then $a(\varphi) = d$ for all $\varphi \in \mathrm{IBr}_p(B)$.*

Proof. By Brauer's height zero conjecture which has an affirmative answer for p-solvable groups due to [5], the heights of all irreducible complex characters in B are zero. Thus the heights of all irreducible Brauer characters in B are zero since they are liftable by the Fong-Swan Theorem. The assertion now follows by ([8], Theorem 2.1). □

Example 7.3. Let J_1 denote the first simple Janko group. Recall that the Sylow 2-subgroup of G is elementary abelian of order 8. The principal 2-block of G contains 5 irreducible Brauer characters of degree $1, 20, 56, 56, 76$. The heights are $0, 2, 3, 3, 2$. For $a(\varphi)$ one computes $3, 1, 1, 1, 1$. Thus the above Proposition does not hold in general.

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