A Lower Bound on the Weight Hierarchies of Product Codes

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Abstract

The weight of a code is the number of coordinate positions where not all codewords are zero. The rth minimum weight d_r is the least weight of an r-dimensional subcode. Wei and Yang conjectured a formula for the minimum weights of some product codes, and this conjecture has recently been proved in two different ways. In this self-contained paper we give a further generalisation, with a new proof which also covers the old results.

Key words: product code, projective multiset, weight hierarchy, Segre embedding, chain condition

1 Introduction

Generalised Hamming weights have received a lot of attention after Victor Wei's paper [10] in 1991. An early project by Victor Wei and Kyeongcheol Yang [11] started determining the weight hierarchy of product codes, given the weight hierarchies for the component codes.

In the special case where both the component codes satisfy the chain condition they found an upper bound on the weight hierarchy. They conjectured that this bound is always satisfied with equality.

Two different proofs of the conjecture have appeared recently [9,7]. Each of the proofs gives interesting generalisations of the Wei-Yang conjecture. In this paper we give a further generalisation of these results. In the appendix, we suggest some open problems for future study.

1.1 Basic Notation

Let C be a linear $[n,k]$ code over a finite field $\mathbb{F}_q$. If S is a subset of $\mathbb{F}_q^n$, let $\chi(S)$ denote the set $\chi(S) = \bigcup_{s\in S}\chi(s)$, where $\chi(s) = \{i \mid s_i \neq 0\}$ for $s = (s_1, s_2, \ldots, s_n)$. The weight hierarchy of C is the sequence $(d_1(C), \ldots, d_k(C))$, where

$$d_r(C) := \min\{\#\chi(D) \mid D \leqq C, \dim D = r\}.$$

The weight hierarchy is an increasing sequence with $d_1(C)$ the minimum Hamming distance. We call C a chained code if there exists a chain

$$\{0\} = D_0 < D_1 < \ldots < D_k = C,$$

such that D_i has dimension i and weight $d_i(C)$.

We consider now the tensor product $C = C_1 \otimes \ldots \otimes C_t$ of $[n_i, k_i]$ codes C_i. Clearly, C is an $[n,k]$ code, where $n = n_1 n_2 \ldots n_t$ and $k = k_1 k_2 \ldots k_t$. What can we tell about the weight hierarchy of C from the knowledge of the weight hierarchies of the C_i?

2 The Main Result

To state the main result, we need some definitions. First let

$$\mathcal{M}_t := \{\mathbf{i} = (i_1, i_2, \ldots, i_{t-1}) \mid 1 \leq i_j \leq k_j, 1 \leq j < t\}.$$

Definition 1 *Let π be a map $\mathcal{M}_t \to \{0, 1, \ldots, k_t\}$ given by $\mathbf{i} \mapsto t_{\mathbf{i}}$. We call π a $(k_1, k_2, \ldots, k_t)$-partition of r if*

(1) $\sum_{\mathbf{i}\in\mathcal{M}_t} t_{\mathbf{i}} = r$, and
(2) π is a decreasing function in each coordinate, i.e.

$$t_{i_1,\ldots,i_j,\ldots,i_{t-1}} \leq t_{i_1,\ldots,i_j-1,\ldots,i_{t-1}}$$

for $j = 1, \ldots, t-1$ and $1 < i_j$.

Let $\mathcal{P}(k_1, k_2, \ldots, k_t; r)$ denote the set of all $(k_1, k_2, \ldots, k_t)$-partitions of r. For $\pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$, we define

$$\nabla(\pi) := \sum_{\mathbf{i}\in\mathcal{M}_t} \prod_{j=1}^{t-1} (d_{i_j}(C_j) - d_{i_j-1}(C_j)) d_{\pi(\mathbf{i})}(C_t). \tag{1}$$

Note that $\nabla(\pi)$ depends on the weight hierarchies of all the codes C_i. Now let

$$d_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t) := \min\left\{\nabla(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)\right\},$$

for $r = 1, 2, \ldots, k$. This number was first defined in [11] for $t = 2$. It was generalised to arbitrary t in [7].

Theorem 2 *Let $C = C_1 \otimes C_2 \otimes \ldots \otimes C_t$ be the product of linear codes C_i. Then $d_r(C) \geq d_r^*(C)$ for all $r = 1, 2, \ldots, k$. Moreover, equality holds if all the components C_i are chained.*

The Wei-Yang Conjecture [11] said that $d_r^*(C_1 \otimes C_2) = d_r(C_1 \otimes C_2)$ if C_1 and C_2 are chained. Several researchers worked with this problem throughout the nineties. Barbero and Tena [1] worked with $r \leq 4$, and found some general results, which also coincide with the Wei-Yang conjecture when the component codes are chained. The number $d_r^*(C)$ was computed for certain classes of codes in [4,8], and the results in [8] turned out to verify the Wei-Yang conjecture in the appropriate special cases.

The case $t = 2$ of Theorem 2, and hence the Wei-Yang Conjecture, was proved in [9]. The generalisation for products of chained codes was performed in [7]. Thus only the upper bound remains to be proved.

We will give a complete proof of the entire theorem, using the techniques from [9]. This technique is very different from the one applied in [7].

Some readers will miss some examples of computing d_r^*. None is included here, but the interested reader may find some in [9]. It should also be observed that when the component codes are not chained, we may have $d_r \neq d_r^*$, a fact which is also demonstrated by example in [9].

3 Proof of the Theorem

3.1 Projective Multisets

We shall prove the theorem in the language of projective multisets, which naturally arise in coding theory by considering the columns of some generator matrix as representatives of points in the projective space $\mathbb{P}^{k-1}$. It is customary to assume that the code has no zero-positions. We make this assumption as well, but all the results are valid for codes with zero-positions.

For a projective multiset γ and a point $x \in \mathbb{P}^{k-1}$, we let $\gamma(x)$ denote the multiplicity (or value) of x in γ. This definition is extended to $S \subset \mathbb{P}^{k-1}$ by setting

$$\gamma(S) := \sum_{s \in S} \gamma(s).$$

Instead of using the weights, it is more convenient to deal with the dual weights $\Delta_i(C)$ which we define by

$$\Delta_i(C) := n - d_{k-i-1}(C) = \sum_{j=0}^{i} \delta_j(C),$$

where

$$\delta_i(C) := d_{k-i}(C) - d_{k-i-1}(C).$$

Analogously, we write

$$\Delta_i^*(C) := n - d_{k-i-1}^*(C).$$

Thus, in order to prove the theorem, we will prove that $\Delta_r(C) \leq \Delta_r^*(C)$ for $r = 0, 1, \ldots, k-1$.

In [6], it was shown that there is a one-to-one correspondence between subcodes $D \leqq C$ and subspaces $\Pi \leqq \mathbb{P}^{k-1}$, such that $\dim \Pi = k - 1 - \dim D$ and $\gamma(\Pi) = n - w(D)$, where γ is the projective multiset obtained by taking the columns of some generator matrix of C. In particular, $\Delta_r(C) = n - d_{k-1-r}(C)$ is the maximum value of any r-space $\Pi \leqq \mathbb{P}^{k-1}$. This is the fundamental lemma for studying higher weights in the language of projective multisets.

The following lemma should be fairly easy to verify by investigating the generator matrices, but a complete proof may be found in [9].

Lemma 3 *If γ_1 and γ_2 are projective multisets corresponding to C_1 and C_2, then the projective multiset corresponding to $C_1 \otimes C_2$ is formed by the image of $\gamma_1 \times \gamma_2$ under the Segre embedding.*

The Segre embedding of $\mathbb{P}^{k_1-1} \times \mathbb{P}^{k_2-1}$ in $\mathbb{P}^{k_1 k_2 - 1}$ is given by

$$((x_1, x_2, \ldots, x_{k_1}), (y_1, y_2, \ldots, y_{k_2})) \mapsto (x_i y_j \mid 1 \leq i \leq k_1, 1 \leq j \leq k_2).$$

This map is well-known in algebraic geometry [3]. It is an injective morphism, and its image is a subvariety of $\mathbb{P}^{k_1 k_2 - 1}$, called the Segre variety.

Definition 4 (Dual partition) *For every $\pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$, the dual partition is defined as*

$$\pi^*(\mathbf{i}) := k_t - \pi((k_1 + 1, k_2 + 1, \ldots, k_{t-1} + 1) - \mathbf{i}).$$

Note that $\pi^* \in \mathcal{P}(k_1, k_2, \ldots, k_t; k-r)$ and $(\pi^*)^* = \pi$. Furthermore, we define

$$\Delta(\pi) := n - \nabla(\pi^*).$$

With this notation, we get

$$\begin{aligned}\Delta_r^*(C) &= n - d_{k-1-r}^*(C)\\ &= n - \min\{\nabla(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; k-1-r)\}\\ &= \max\{\Delta(\pi^*) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; k-1-r)\}.\end{aligned}$$

Hence

$$\Delta_r^*(C) = \max\{\Delta(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)\}.$$

The next step is to derive a more accessible expression for $\Delta(\pi)$.

Definition 5 *Let $\pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$, and take $s \in \{1, 2, \ldots, k_1\}$. The s-th subpartition $\pi|_s$ of π is given by*

$$\pi|_s(i_2, i_3, \ldots, i_{t-1}) = \pi(s, i_2, i_3, \ldots, i_{t-1}).$$

Clearly $\pi|_s \in \mathcal{P}(k_2, k_3, \ldots, k_t; r_s)$ for some integer r_s, and $r_1 + r_2 + \ldots + r_{k_1} = r$. Define also

$$\mathcal{M}_t^1 := \{\mathbf{i} = (i_2, i_3, \ldots, i_{t-1}) \mid 1 \le i_j \le k_j, 1 < j < t\}.$$

Lemma 6 *The dual $(\pi|_s)^*$ of $\pi|_s$ is the $(k_1 - s + 1)$-th subpartition $\pi^*|_{k_1-s+1}$ of π^*.*

PROOF. We have

$$\begin{aligned}(\pi|_s)^*(\mathbf{i}) &= k_t - \pi|_s(k_2 + 1 - i_2, k_3 + 1 - i_3, \ldots, k_{t-1} + 1 - i_{t-1})\\ &= k_t - \pi(s, k_2 + 1 - i_2, k_3 + 1 - i_3, \ldots, k_{t-1} + 1 - i_{t-1}).\end{aligned}$$

Then we have

$$\begin{aligned}\pi^*|_{k_1-s+1}(\mathbf{i}) &= \pi^*(k_1 - s + 1, i_2, i_3, \ldots, i_{t-1})\\ &= k_t - \pi(s, k_2 + 1 - i_2, k_3 + 1 - i_3, \ldots, k_{t-1} + 1 - i_{t-1}).\end{aligned}$$

□

Lemma 7 *We have*

$$\Delta(\pi) = \sum_{\mathbf{i} \in \mathcal{M}_t} \Delta_{t_{\mathbf{i}}-1}(C_t) \prod_{j=1}^{t-1} \delta_{i_j-1}(C_j),$$

for all $\pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$ *and* $0 \leq r \leq k$.

PROOF. The proof runs by induction on t. The lemma was proved for $t = 2$ in [9], so assume it holds for $t - 1$.

Observe that $\Delta(\pi) = n - \nabla(\pi^*)$. The definition of $\nabla(\pi^*)$ from (1) is

$$\nabla(\pi^*) := \sum_{\mathbf{i} \in \mathcal{M}_t} \prod_{j=1}^{t-1} (d_{i_j}(C_j) - d_{i_j - 1}(C_j)) d_{\pi^*(\mathbf{i})}(C_t).$$

Thus we get

$$\begin{aligned}
\nabla(\pi^*) &= \sum_{i_1=1}^{k_1} \left[\sum_{\mathbf{i} \in \mathcal{M}_t^1} d_{\pi^*|_{i_1}(\mathbf{i})}(C_t) \prod_{j=2}^{t-1} \delta_{k_j - i_j}(C_j) \right] \delta_{k_1 - i_1}(C_1) \\
&= \sum_{i_1=1}^{k_1} \left[\sum_{\mathbf{i} \in \mathcal{M}_t^1} d_{(\pi|_{k_1+1-i_1})^*(\mathbf{i})}(C_t) \prod_{j=2}^{t-1} \delta_{k_j - i_j}(C_j) \right] \delta_{k_1 - i_1}(C_1) \\
&= \sum_{i_1=1}^{k_1} \left[n_2 \cdot n_3 \cdot \ldots \cdot n_t - \Delta(\pi|_{k_1+1-i_1}) \right] \delta_{k_1 - i_1}(C_1) \\
&= n - \sum_{i_1=1}^{k_1} \Delta(\pi|_{k_1+1-i_1}) \delta_{k_1 - i_1}(C_1).
\end{aligned}$$

Hence

$$\Delta(\pi) = \sum_{i_1=1}^{k_1} \Delta(\pi|_{k_1+1-i_1}) \delta_{k_1 - i_1}(C_1) = \sum_{i_1=1}^{k_1} \Delta(\pi|_{i_1}) \delta_{i_1 - 1}(C_1). \tag{2}$$

By the induction hypothesis, we get

$$\begin{aligned}
\Delta(\pi) &= \sum_{i_1=1}^{k_1} \left[\sum_{\mathbf{i} \in \mathcal{M}_t^1} \Delta_{t_{\mathbf{i}} - 1}(C_t) \prod_{j=2}^{t-1} \delta_{i_j - 1}(C_j) \right] \delta_{i_1 - 1}(C_1). \\
&= \sum_{\mathbf{i} \in \mathcal{M}_t} \Delta_{t_{\mathbf{i}} - 1}(C_t) \prod_{j=1}^{t-1} \delta_{i_j - 1}(C_j),
\end{aligned}$$

as required. $\square$

Define

$$\hat{\mathcal{P}}(k_1, k_2, \ldots, k_t; r) := \bigcup_{r' \leq r} \mathcal{P}(k_1, k_2, \ldots, k_t; r').$$

We have a partial ordering on $\hat{\mathcal{P}}(k_1, k_2, \ldots, k_t; r)$ by setting $\pi' \leq \pi$ if $\pi'(\mathbf{i}) \leq \pi(\mathbf{i})$ for all $\mathbf{i} \in \mathcal{M}_t$. If $\pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$ is a partition, write $\Sigma\pi = r$ for the sum of its values.

Note that if we have a sequence of $(k_2, k_3, \ldots, k_t)$-partitions $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_{k_1}$, then the π_i define the subpartitions $\pi|_i$ of some partition $\pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$, where $r = \Sigma\pi_1 + \Sigma\pi_2 + \ldots + \Sigma\pi_{k_1}$.

3.2 The simple case

In this section we study the product codes of two components. We will continue by induction on t in the next section. The proof here is slightly different from the proof given for $t = 2$ in [9].

Let $C = C_1 \otimes C_2$, where $\dim C = k$. Let γ_1, γ_2, and γ be the projective multiset corresponding to C_1, C_2, and C respectively.

Let $\Pi \leqq \mathbb{P}^{k-1}$. We will define the *associated partition* $\pi(\Pi)$ of Π. For $0 \leq i \leq k_1 - 1$, let Θ_i be the set of points $b \in \mathbb{P}^{k_2-1}$ such that there is an i-space $\Phi_i \leqq \mathbb{P}^{k_1-1}$ with $\Phi_i \otimes b \subseteq \Pi$. Let $\pi(\Pi)(i) = \dim\langle\Theta_{i-1}\rangle + 1$ for $i = 1, 2, \ldots, k_1$. Obviously $\Theta_i \subseteq \Theta_{i-1}$. Hence $\pi(\Pi)$ is indeed a partition. Finally put $t_i := \dim\langle\Theta_i\rangle$.

For $x \in \mathbb{P}^{k_2-1}$ define

$$R(x) := \{p \otimes x \in \Pi \mid p \in \mathbb{P}^{k_1-1}\}.$$

By the bilinearity of the Segre embedding we have $R(x) \leqq \mathbb{P}^{k-1}$.

Lemma 8 *If $\Pi \leqq \mathbb{P}^{k-1}$ is an r-space, then $\pi(\Pi) \in \hat{\mathcal{P}}(k_1, k_2; r + 1)$.*

PROOF. Let $b_0, b_1, \ldots, b_{k_2-1}$ be a basis for $\mathbb{P}^{k_2-1}$ such that $b_0, b_1, \ldots, b_{t_i} \in \Theta_i$. For each i where $0 \leq i < k_2$, let $b_i^0, b_i^1, \ldots, b_i^{r_i}$ be a basis for $R(b_i)$ where $r_i = \max\{j \mid i \leq t_j\}$. Clearly $b_i^j = a_i^j \otimes b_i$ for some $a_i^j \in \mathbb{P}^{k_1-1}$.

Consider the set

$$B := \{b_i^j \mid 0 \leq j \leq k_1 - 1, 0 \leq i \leq t_j\}.$$

Clearly

$$\#B = \sum_{i=0}^{k_1-1} (t_i + 1).$$

The set B is a set of projectively independent points, and $B \subseteq \Pi$. Since $\dim \Pi = r$, we get

$$\Sigma\pi(\Pi) = \sum_{i=0}^{k_1-1} (t_i + 1) = \#B \leq r + 1,$$

proving the lemma. $\square$

Lemma 9 *If* $\Pi \leqq \mathbb{P}^{k-1}$*, then* $\gamma(\Pi) \leq \Delta(\pi(\Pi))$.

PROOF. For convenience, we write $\Theta_{k_1} := \emptyset$. If $b \in \Theta_i \backslash \Theta_{i+1}$, then $R(b) = \Phi_i(b) \otimes b$, where $\Phi_i(b)$ is an i-space in $\mathbb{P}^{k_1-1}$.

By Lemma 3, we have

$$\Pi = R(\Theta_0) = \bigcup_{b \in \Theta_0} R(b) = \bigcup_{i=0}^{k_1-1} \bigcup_{b \in \Theta_i \backslash \Theta_{i+1}} (\Phi_i(b) \otimes b).$$

Note that the union is disjoint. Hence

$$\begin{aligned}
\gamma(\Pi) = \gamma\left(\bigcup_{b\in\Theta_0} R(b)\right) &= \sum_{i=0}^{k_1-1} \sum_{b \in \Theta_i \backslash \Theta_{i+1}} \gamma_1(\Phi_i(b)) \cdot \gamma_2(b) \\
&\leq \sum_{i=0}^{k_1-1} \sum_{b \in \Theta_i \backslash \Theta_{i+1}} \Delta_i(C_1) \cdot \gamma_2(b) = \sum_{i=0}^{k_1-1} \delta_i(C_1) \sum_{b \in \Theta_i} \gamma_2(b) \\
&\leq \sum_{i=0}^{k_1-1} \delta_i(C_1) \Delta_{t_i}(C_2) = \sum_{i=0}^{k_1-1} \delta_i(C_1) \Delta_{\pi(i+1)-1}(C_2) \\
&= \sum_{i=1}^{k_1} \delta_{i-1}(C_1) \Delta_{\pi(i)-1}(C_2) = \Delta(\pi(\Pi)).
\end{aligned} \tag{3}$$

Thus the lemma is proved. $\square$

We observe that this lemma implies $\Delta_r(C_1 \otimes C_2) \leq \Delta_r^*(C_1 \otimes C_2)$ and thus proves the bound from Theorem 2 for $t = 2$.

Lemma 10 *If* $\Pi' \leqq \Pi \leqq \mathbb{P}^{k-1}$*, then* $\pi(\Pi') \leq \pi(\Pi)$.

PROOF. Let Θ_i and t_i be as in the definition of $\pi(\Pi)$, and let Θ_i' and t_i' be the corresponding objects for Π'. We only have to prove that $t_i' \leq t_i$ for all i. We obtain Π' from Π by removing points. Hence $\Theta_i' \subseteq \Theta_i$ for all i, and $t_i' \leq t_i$ as required. $\square$

3.3 The general case

The t component codes C_i correspond to t projective multisets γ_i on $\mathbb{P}^{k_i-1}$ for $i = 1, 2, \ldots, t$. Let γ be the multiset corresponding to $C = C_1 \otimes C_2 \otimes \ldots \otimes C_t$, and let $k := \dim C$.

Lemma 11 *For every subspace* $\Pi \leqq \mathbb{P}^{k-1}$ *of dimension* r *there is a well-defined associated partition* $\pi(\Pi) \in \hat{\mathcal{P}}(k_1, k_2, \ldots, k_t; r+1)$ *such that*

(a) $\gamma(\Pi) \leq \Delta(\pi(\Pi))$*; and*
(b) if $\Pi' \leqq \Pi \leqq \mathbb{P}^{k-1}$*, then* $\pi(\Pi') \leq \pi(\Pi)$.

PROOF. We argue by induction on t. The base case, $t = 2$, is proved in Lemmata 8, 9, and 10. Write $k' := k_2 \cdot k_3 \cdot \ldots \cdot k_t$. Let γ' be the projective multiset corresponding to $C_2 \otimes C_3 \otimes \ldots \otimes C_t$.

Let $\Theta_i \subseteq \mathbb{P}^{k'-1}$ be the set of points p such that there exists an i-space $\Phi_i \leqq \mathbb{P}^{k_1-1}$ where $\Phi_i \otimes p \subseteq \Pi$. Obviously $\Theta_i \subseteq \Theta_{i-1}$. Let $t_i := \dim\langle\Theta_i\rangle$.

By the inductive hypothesis (b) there is an (unique) associated partition $\pi_i \in \hat{\mathcal{P}}(k_2, k_3, \ldots, k_t; t_i + 1)$ to $\langle\Theta_i\rangle$ such that

$$\gamma'(\Theta_i) \leq \gamma'(\langle\Theta_i\rangle) \leq \Delta(\pi_i) \tag{4}$$

for each i. Furthermore $\pi_i \leq \pi_{i-1}$ by the inductive hypothesis (2) since $\Theta_i \subseteq \Theta_{i-1}$. Hence the π_i can be viewed as the k_1 subpartitions of some partition $\pi \in \hat{\mathcal{P}}(k_1, k_2, \ldots, k_t; r'+1)$ where

$$r' := \sum_{i=0}^{k_1-1}(t_i + 1) - 1.$$

More precisely, $\pi(i_1, i_2, \ldots, i_{t-1}) = \pi_{i_1-1}(i_2, i_3, \ldots, i_{t-1})$. By an argument similar to that in the proof of Lemma 8, we get that $r' \leq r$. Hence

$$\pi \in \hat{\mathcal{P}}(k_1, k_2, \ldots, k_t; r+1).$$

For $x \in \mathbb{P}^{k'-1}$ define

$$R(x) := \{p \otimes x \in \Pi \mid p \in \mathbb{P}^{k_1-1}\}.$$

By the bilinearity of the Segre embedding, $R(x) \leqq \mathbb{P}^{k-1}$. If $b \in \Theta_i \backslash \Theta_{i+1}$, then $R(b) = \Phi_i(b) \otimes b$ for some i-space $\Phi_i(b) \in \mathbb{P}^{k_1-1}$.

Now we can write (as in the proof of Lemma 9),

$$\Pi = R(\Theta_0) = \bigcup_{i=0}^{k_1-1} \bigcup_{b \in \Theta_i \backslash \Theta_{i+1}} R(b).$$

Hence we get

$$\begin{aligned}\gamma(\Pi) &= \sum_{i=0}^{k_1-1} \sum_{b\in\Theta_i\setminus\Theta_{i+1}} \gamma_1(\Phi_i(b))\gamma'(b) \\ &\leq \sum_{i=0}^{k_1-1} \Delta_i(C_1) \sum_{b\in\Theta_i\setminus\Theta_{i+1}} \gamma'(b) = \sum_{i=0}^{k_1-1} \delta_i(C_1) \sum_{b\in\Theta_i} \gamma'(b) \\ &= \sum_{i=0}^{k_1-1} \delta_i(C_1)\gamma'(\Theta_i) \leq \sum_{i=0}^{k_1-1} \delta_i(C_1)\Delta(\pi_i) = \Delta(\pi).\end{aligned} \tag{5}$$

The bound in the last line follows from (4), and the very last equality follows from (2). This proves (a) assuming that (a) and (b) holds for $t-1$. It remains to prove that (b) holds.

Let π' be the partition associated with Π', and let $\pi'_i := \pi'|_i$ be the associated subpartitions. To prove the second statement of the lemma, it is sufficient to show that $\pi'_i \leq \pi_i$ for all i.

Let Θ'_i be the Θ_i related to Π'. Recall that π'_i is the partition associated with $\langle\Theta'_i\rangle$. We obtain Π' by removing points from Π. Hence $\langle\Theta'_i\rangle \leqq \langle\Theta_i\rangle$, and by the inductive hypothesis $\pi'_i \leq \pi_i$ as required. □

3.4 When the Chain Condition holds

Lemma 12 *If $C_1, C_2, \ldots, C_t$ satisfy the chain condition, then for every*

$$\pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)$$

there is an r-space $\Pi \leqq \mathbb{P}^{k-1}$ such that $\pi(\Pi) = \pi$ and $\Delta(\pi) = \gamma(\Pi)$.

Moreover, if $\pi' \in \mathcal{P}(k_1, k_2, \ldots, k_t; r'+1)$ and $\pi' \leq \pi$, then there is an r'-space $\Pi' \leqq \Pi$ such that $\pi(\Pi') = \pi'$ and $\Delta(\pi') = \gamma(\Pi')$.

PROOF. First consider the case where $t=2$. Let $p_0, p_1, \ldots, p_{k_1-1}$ be a basis for $\mathbb{P}^{k_1-1}$ such that $\langle p_0, p_1, \ldots, p_i\rangle$ is an i-space of maximum value for C_1. Let

$$\emptyset = \Phi_{-1} \subset \Phi_0 \subset \ldots \subset \Phi_{k_2-1} = \mathbb{P}^{k_2-1}$$

be a chain of subspaces of maximum value for C_2. Write $t_i = \pi(i+1) - 1$ for $i = 0, 1, \ldots, k_1 - 1$, and let $\Pi = \langle p_i \otimes \Phi_{t_i} \mid i = 0, 1, \ldots, k_1 - 1\rangle$. Clearly $\dim \Pi = r$ and $\pi(\Pi) = \pi$. As for the value, it is not hard to verify equality in (3). This proves the first statement of the lemma for $t = 2$.

Let $t'_i = \pi'(i+1) - 1$ for $i = 0, 1, \ldots, k_1 - 1$, and let $\Pi' = \langle p_i \otimes \Phi_{t'_i} \mid i = 0, 1, \ldots, k_1 - 1 \rangle$. Clearly $\Pi' \subseteq \Pi$, and the remaining properties of Π' follows by the argument above. Hence the lemma is proved for $t = 2$.

Assuming that the lemma holds for $t-1$, the inductive step is similar. Let $k' = k_2 k_3 \ldots k_t$, and γ' the projective multiset corresponding to $C' = C_2 \otimes \ldots \otimes C_t$. Let $p_0, p_1, \ldots, p_{k_1-1}$ be a basis for $\mathbb{P}^{k_1-1}$ such that $\langle p_0, p_1, \ldots, p_i \rangle$ is an i-space of maximum value for C_1. Let $\pi_i = \pi|_{i+1}$ be the subpartitions of π. By the inductive hypothesis, there is a chain

$$\Phi(\pi_{k_1}) \subseteq \Phi(\pi_{k_1-1}) \subseteq \ldots \subseteq \Phi(\pi_1) \subseteq \mathbb{P}^{k_1-1},$$

of subspaces of value $\gamma'(\Phi(\pi_i)) = \Delta'(\pi_i)$, where $\Delta'(\pi)$ is computed with the weight hierarchy of C'. We define

$$\Pi = \langle p_i \otimes \Phi(\pi_i) \mid i = 0, 1, \ldots, k_1 - 1 \rangle.$$

Clearly $\dim \Pi = \sum_{i=1}^{k_1} \Sigma \pi_i - 1 = r$. The value is given by (5), and it may be verified that equality holds.

Consider at last a partition $\pi' \leq \pi$. We construct as above a chain of subspaces

$$\Phi(\pi'_{k_1}) \subseteq \Phi(\pi'_{k_1-1}) \subseteq \ldots \subseteq \Phi(\pi'_1) \subseteq \mathbb{P}^{k_1-1}.$$

This can, by the induction hypothesis, be done such that $\Phi(\pi'_i) \subseteq \Phi(\pi_i)$. We define

$$\Pi' = \langle p_i \otimes \Phi(\pi'_i) \mid i = 0, 1, \ldots, k_1 - 1 \rangle.$$

Clearly $\Pi' \subseteq \Pi$, and the remaining properties are proved as in the previous paragraph. $\square$

A Some Future Work

The support weight distribution was introduced in [5]. It was proved that if the support weight distribution for C is known, then one can also find the weight distribution of $C \otimes S$, where S is a simplex code. The support weight distribution is only known for a very few classes of codes. Is it possible to find the support weight distribution of some classes of product codes, such as the product of two simplex codes?

David Forney [2] proved that the generalised Hamming weights give a lower bound on the state complexity of a minimal trellis. It was proved that this bound is met with equality with some optimal bit ordering if and only if the code meets the so-called two-way chain condition. Is it possible to determine completely the state complexity of a product code, given the state complexities of the component codes?

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