On the weight hierarchy of product codes

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Abstract **Let V and W be codes and let $C = V \otimes W$ be the product code of V and W. In [6], Wei and Yang conjectured a formula for the generalized Hamming weights of $V \otimes W$ in terms of those of V and W provided that both V and W satisfy the chain condition. Recently the conjecture has been proved by Schaathun in [4]. In this paper we generalize the formula to a product code with more than two components.**

Keywords **Product code, weight hierarchy, chain condition.**

I. INTRODUCTION

Throughout the paper all codes are linear and defined over the finite field K.
For $v = (x_1, \dots, x_n) \in K^n$ we define the support of v by

$$\text{supp } v = \{i \mid x_i \neq 0\}$$

and for $X \subseteq K^n$ we put

$$\text{supp } X = \bigcup_{v \in X} \text{supp } v.$$

If D is a subcode of the linear code C we write $D \leq C$ and $D < C$ in case D is a proper subspace of C. The weight hierarchy of an $[n, k]$ code C is the sequence $(d_1, \dots, d_k)$ where

$$d_r = d_r(C) = \min\{|\operatorname{supp} D| \mid D \leq C, \dim D = r\}.$$

In particular $d_1 = d$. For convenience we also set $d_0 = 0$. One says that C satisfies the chain condition if there exists a sequence

$$C_1 < C_2 < \cdots < C_k = C$$

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of subspaces C_r of C with $|\operatorname{supp} C_r| = d_r$ for $r = 1, \ldots, k$. Note that many good codes satisfy the chain condition; for instance, Hamming codes, Simplex codes, Reed-Muller codes, MDS codes, extended Golay codes (see [6]). But, and that is more of theoretical interest, most codes do not satisfy the chain condition (see [2]).

Now let V^i be $[n_i, k_i]$ codes for $i = 1, \ldots, t$. The product code C of the V^i's is defined as the tensor product $V^1 \otimes \ldots \otimes V^t$. The problem in which we are interested is the following: *Can one compute the weight hierarchy of C from knowledge of the weight hierarchies of the components V^i?*

For $t = 2$ Wei and Yang defined in [6] the expression

$$d_r^*(V^1 \otimes V^2) = \min \sum_{i=1}^{k_1} (d_i(V^1) - d_{i-1}(V^1)) d_{t_i}(V^2)$$

where the minimum is taken over all sequences (t_i) with $0 \leq t_{k_1} \leq \cdots \leq t_2 \leq t_1 \leq k_2$ and $\sum_{i=1}^{k_1} t_i = r \leq k_1 k_2$.

In the same paper the authors proved that

$$d_r(V^1 \otimes V^2) \leq d_r^*(V^1 \otimes V^2)$$

for all $r = 1, \ldots, k_1 k_2$ if both V^1 and V^2 satisfy the chain condition and they conjectured that even equality holds true.

For small r's, i.e. $r \leq 4$, the conjecture was verified by Barbero and Tena in [1] and recently in full generality by Schaathun in [4]. More precisely he proved that

$$d_r^*(V^1 \otimes V^2) \leq d_r(V^1 \otimes V^2)$$

always holds true, i.e., without assuming any chain condition. Furthermore, in [3], Helleseth and Kløve computed d_r^* for the tensor product of some codes which satisfy the chain condition. Thus their formula for d_r^* describe exactly the weight hierarchy.

In this paper we generalize the Wei-Yang formula to a product code with an arbitrary number t of components. In order to do that we need the following definition.

Let t, r and k_i $(i \in \{1, \ldots, t\})$ be positive integers. We put

$$\mathcal{M}_t := \{\underline{i} = (i_1, \ldots, i_{t-1}) \mid 1 \leq i_j \leq k_j, 1 \leq j \leq t-1\}$$

for $t > 1$ and $\mathcal{M}_1 = \{0\}$. A function $\tau\colon \mathcal{M}_t \to \{0, 1, \dots, k_t\}$ is called a $(k_1, k_2, \dots, k_t)$-partition of r if

(i) $\Sigma_{\underline{i} \in \mathcal{M}_t} \tau(\underline{i}) = r$ and

(ii) τ is a decreasing function in each coordinate, i. e.
$\tau((i_1, \dots, i_j, \dots, i_{t-1})) \leq \tau((i_1, \dots, i_j - 1, \dots, i_{t-1}))$ for $j = 1, \dots, t-1$ and $i_j > 1$.

Now let V^i be $[n_i, k_i]$ codes for $i = 1, \dots, t$.
For a $(k_1, \dots, k_t)$-partition τ of r we define

$$\nabla(\tau) = \Sigma_{\underline{i} \in \mathcal{M}_t} \Pi_{j=1}^{t-1} (d_{i_j}(V^j) - d_{i_j - 1}(V^j)) d_{\tau(\underline{i})}(V^t).$$

Note that this formula specializes to the Wei-Yang formula for $t = 2$.
With the notation above we can now state our main result.
Theorem If each of the codes $V^1, \dots, V^t$ satisfies the chain condition, then

$$d_r(V^1 \otimes \ldots \otimes V^t) = \min \{\nabla(\tau) \mid \tau \text{ a } (k_1, \dots, k_t)\text{-partition of } r\}.$$

This result and [4] finally led to

$$d_r(V^1 \otimes \ldots \otimes V^t) \geq \min \{\nabla(\tau) \mid \tau \text{ a } (k_1, \dots, k_t)\text{-partition of } r\}$$

for arbitrary codes V^i, i.e., for codes V^i which do not necessarily satisfy the chain condition. A proof will be provided in a forthcoming paper of Schaathun and the second author ([5]).

In order to prove the Theorem one might try to use induction based on the affirmative answer to the Wei-Yang conjecture. It turns out that such an approach does not apply since the product code of two codes satisfying the chain condition usually fails to satisfy the chain condition as the example at the end of this paper shows.

II. PRELIMINARIES

First, we prove some results about product codes with two components which we will need in the next section.

Let $C = V \otimes W$ denote the tensor product of V and W where V is an $[n,k]$ code and W is an $[m,l]$ code. If

$$(y_1,\dots,y_m)^t(x_1,\dots,x_n) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} (x_1,\dots,x_n) = (y_i x_j)_{\substack{i=1,..,m \\ j=1,..,n}}$$

denotes the dyadic product of $w = (y_1,\dots,y_m)$ and $v = (x_1,\dots,x_n)$, then C can be represented by

$$C = \langle w^t v \mid v \in V, w \in W \rangle,$$

i.e., the elements of C are matrices of type (m,n), where the rows are elements of V and the columns are elements of W.

For $x \in C$ we denote by $\pi_j^C(x)$ $(j = 1,\dots,n)$ the j-th column and by $\pi_i^R(x)$ $(i = 1,\dots,m)$ the i-th row of x. If D is a subset of C we put

$$\pi_j^C(D) = \{\pi_j^C(x) \mid x \in D\} \subseteq W \text{ and}$$
$$\pi_i^R(D) = \{\pi_i^R(x) \mid x \in D\} \subseteq V.$$

Furthermore, let

$$\rho_j^C(D) = \{x \in D \mid \pi_j^C(x) = 0\} \text{ and}$$
$$\rho_i^R(D) = \{x \in D \mid \pi_i^R(x) = 0\}.$$

For any vector v and $1 \le s \le \mathrm{length}(v)$ we denote by $v_{(s)}$ the s-th coordinate of v.

Lemma 1. Let $C = V \otimes W$. If $W_{(s)} = \{w \in W \mid w_{(s)} = 0\}$ then $\rho_s^R(C) = V \otimes W_{(s)}$.

Proof. If $W_{(s)} = W$, the assertion is obvious. So we may assume that $W = \langle w \rangle \oplus W_{(s)}$ for some $w \in W \setminus W_{(s)}$. Let $x \in \rho_s^R(C)$. Since $V \otimes W = V \otimes \langle w \rangle \oplus V \otimes W_{(s)}$, we may write $x = y + z$ with $y = v \otimes w$ $(v \in V)$ and $z \in V \otimes W_{(s)}$. Obviously $\pi_s^R(V \otimes W_{(s)}) = 0$, hence

$$0 = \pi_s^R(x) = \pi_s^R(y) \oplus \pi_s^R(z) = \pi_s^R(y) = \pi_s^R(v \otimes w).$$

Now, as $w \notin W_{(s)}$ we obtain $v = 0$, therefore $x = z \in V \otimes W_{(s)}$ and the assertion of the lemma follows since $V \otimes W_{(s)} \le \rho_s^R(C)$. **q.e.d.**

From now on we suppose that V satisfies the chain condition, i.e., there exists a chain $V_1 < V_2 < \ldots < V_k = V$ such that $d_r(V) = \mid \mathrm{supp}\, V_r \mid$ for $r = 1,\dots,k$. We put $V_0 = 0$. No assumption is made w.r.t. chain condition for the space W.

The next result is fundamental in what follows. Essentially, it asserts that, when looking for r-subspaces of $V \otimes W$ of minimum support, we only have to look at subspaces of a very special kind.

Proposition 2. Let $D \leq V \otimes W$. Then there exists a basis of W, say $\mathbf{W} = \{w_1, \ldots, w_l\}$ and integers $0 \leq \alpha_i \leq k$ $(i = 1, \ldots, l)$ such that for $\bar{D} = V_{\alpha_1} \otimes w_1 \oplus \ldots \oplus V_{\alpha_l} \otimes w_l$ the following conditions hold.

$$\begin{array}{ll} (i) & \dim D = \dim \bar{D} \text{ and} \\ (ii) & \mid \operatorname{supp} \pi_i^R(\bar{D}) \mid \leq \mid \operatorname{supp} \pi_i^R(D) \mid \text{ for } 1 \leq i \leq m. \end{array}$$

In particular, we have $\mid \operatorname{supp} \bar{D} \mid \leq \mid \operatorname{supp} D \mid$. Moreover, if $D_1, \ldots, D_h$ are subspaces of $V \otimes W$, and $\bar{D}_i$ is a space obtained from D_i by the previous construction for $i \in \{1, \ldots, h\}$, then $\mid \operatorname{supp}(\bar{D}_1 + \ldots + \bar{D}_h) \mid \leq \mid \operatorname{supp}(D_1 + \ldots + D_h) \mid$.

Proof. First we prove the existence of $\bar{D}$ with the properties (i) and (ii). We argue by induction on l. For $l = 1$ the assertion is obvious. For $i \in \{1, \ldots, m\}$, let $r_i = \dim \pi_i^R(D)$. Furthermore let s be an index such that $r_s = \min\{r_i : i \in \{1, \ldots, m\}, r_i \neq 0\}$. Such an s exists since we may assume that D is not the nullspace.
Since $r_s \neq 0$ and $\dim D = \dim \pi_s^R(D) + \dim \rho_s^R(D) = r_s + \dim \rho_s^R(D)$, we have $\rho_s^R(D) < D$. Thus there exists $x \in D$ with $\pi_s^R(x) \neq 0$. Hence there is a $t \in \{1, \ldots, n\}$ such that $0 \neq (\pi_t^C(x))_{(s)}$ which is the s-th coordinate of the column vector $\pi_t^C(x)$.
Let $w = \pi_t^C(x)$. Clearly, $w \in W \setminus W_{(s)}$ and therefore $W_{(s)} < W$. By Lemma 1, we have $\rho_s^R(C) = V \otimes W_{(s)}$ and we may apply the inductive hypothesis to

$$\rho_s^R(D) \leq \rho_s^R(C) = V \otimes W_{(s)}.$$

We deduce that there exists a basis $\mathbf{W}_{(s)} = \{w_2, \ldots, w_l\}$ of $W_{(s)}$ and integers $0 \leq \alpha_i \leq k$ $(i = 2, \ldots, l)$ such that for

$$\overline{\rho_s^R(D)} = V_{\alpha_2} \otimes w_2 \oplus \ldots \oplus V_{\alpha_l} \otimes w_l$$

we have $\dim \overline{\rho_s^R(D)} = \dim \rho_s^R(D)$ and $\mid \operatorname{supp} \pi_i^R(\overline{\rho_s^R(D)}) \mid \leq \mid \operatorname{supp} \pi_i^R(\rho_s^R(D)) \mid$ for $i \in \{1, \ldots, m\}$.
If $w_1 = w$ which we have fixed above, then $\{w_1, w_2, \ldots, w_l\}$ is a basis of W (note that $w_1 \in W \setminus W_{(s)}$). We put $\alpha_1 = r_s = \dim \pi_s^R(D)$ and

$$\bar{D} = V_{r_s} \otimes w \oplus \overline{\rho_s^R(D)} = V_{\alpha_1} \otimes w_1 \oplus \overline{\rho_s^R(D)}.$$

Thus $\bar{D}$ has the form as desired and

$$\begin{aligned}\dim \bar{D} = \dim V_{r_s} + \dim \overline{\rho_s^R(D)} = r_s + \dim \rho_s^R(D) = \\ = \dim \pi_s^R(D) + \dim \rho_s^R(D) = \dim D.\end{aligned}$$

It remains to show that

$$|\operatorname{supp} \pi_i^R(\bar{D})| \leq |\operatorname{supp} \pi_i^R(D)| \quad \text{for } 1 \leq i \leq m.$$

To prove this we use

$$\pi_i^R(\bar{D}) = \pi_i^R(V_{r_s} \otimes w) + \pi_i^R(\overline{\rho_s^R(D)})$$

and distinguish two cases.

Case $\pi_i^R(D) = 0$: We have

$$w_{(i)} = (\pi_t^C(x))_{(i)} = (\pi_i^R(x))_{(t)} = 0.$$

Thus $\pi_i^R(V_{r_s} \otimes w) = 0$. As

$$0 \leq |\operatorname{supp} \pi_i^R(\overline{\rho_s^R(D)})| \leq |\operatorname{supp} \pi_i^R(\rho_s^R(D))| \leq |\operatorname{supp} \pi_i^R(D)| = 0,$$

we are done.

Case $\pi_i^R(D) \neq 0$: Observe that

$$\begin{aligned}\pi_i^R(\bar{D}) = \pi_i^R(V_{r_s} \otimes w) + \pi_i^R(\overline{\rho_s^R(D)}) = \\ = \pi_i^R(V_{\alpha_1} \otimes w_1) + \pi_i^R(V_{\alpha_2} \otimes w_2) + \ldots + \pi_i^R(V_{\alpha_l} \otimes w_l).\end{aligned}$$

Furthermore note that

$$\operatorname{supp}\ \pi_i^R(V_{\alpha_j} \otimes w_j) = \begin{cases} \operatorname{supp}\ V_{\alpha_j}, & \text{if } (w_j)_{(i)} \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Since the spaces V_{α_j} are members of a chain which defines the chain condition for V, we obtain

$$|\operatorname{supp} \pi_i^R(\bar{D})| = \max \{|\operatorname{supp} \pi_i^R(V_{r_s} \otimes w)|, |\operatorname{supp} \pi_i^R(\overline{\rho_s^R(D)})|\}.$$

Furthermore we have

$$|\operatorname{supp} \pi_i^R(\overline{\rho_s^R(D)})| \leq |\operatorname{supp} \pi_i^R(\rho_s^R(D))| \leq |\operatorname{supp} \pi_i^R(D)|,$$

and since $0 \neq \dim \pi_i^R(D) = r_i \geq r_s$, we also have

$$|\operatorname{supp} \pi_i^R(V_{r_s} \otimes w)| \leq d_{r_s}(V) \leq |\operatorname{supp} \pi_i^R(D)|.$$

From this we deduce

$$|\operatorname{supp}\pi_i^R(\bar{D})| \leq |\operatorname{supp}\pi_i^R(D)|\,.$$

So far we have proved (i) and (ii). Finally we get

$$|\operatorname{supp}\bar{D}| = \Sigma_{i=1}^m |\operatorname{supp}\pi_i^R(\bar{D})| \leq \ \Sigma_{i=1}^m |\operatorname{supp}\pi_i^R(D)| = |\operatorname{supp}D|\,.$$

So we are left with the last claim. We proceed by induction on $\dim D_1 + \ldots + \dim D_h$ to prove that for $i \in \{1, \ldots, m\}$,

$$|\operatorname{supp}\pi_i^R(\bar{D}_1 + \ldots + \bar{D}_h)| \leq |\operatorname{supp}\pi_i^R(D_1 + \ldots + D_h)|\,.$$

Clearly, we may assume that $\dim D_1 \neq 0$. Let $\bar{D}_1 = V_{\alpha_1} \otimes w \oplus \overline{\rho_s^R(D_1)}$ be a space obtained by the previous construction. Note that

$$\dim \rho_s^R(D_1) + \dim D_2 + \ldots + \dim D_h < \dim D_1 + \ldots + \dim D_h.$$

Thus by the induction hypothesis we have

$$|\operatorname{supp}\pi_i^R(\overline{\rho_s^R(D_1)} + \bar{D}_2 + \ldots + \bar{D}_h)| \leq |\operatorname{supp}\pi_i^R(\rho_s^R(D_1) + D_2 + \ldots + D_h)|$$

for $i \in \{1, \ldots, m\}$.
Now, let $i \in \{1, \ldots, m\}$. We distinguish again two cases.
If $\pi_i^R(D_1) = 0$, then $w_{(i)} = 0$. Therefore we obtain

$$\begin{aligned}|\operatorname{supp}\pi_i^R(\bar{D}_1 + \bar{D}_2 + \ldots + \bar{D}_h)| = |\operatorname{supp}\pi_i^R(\overline{\rho_s^R(D_1)} + \bar{D}_2 + \ldots + \bar{D}_h)| \leq \\ |\operatorname{supp}\pi_i^R(\rho_s^R(D_1) + D_2 + \ldots + D_h)| \leq |\operatorname{supp}\pi_i^R(D_1 + \ldots + D_h)|.\end{aligned}$$

If $\pi_i^R(D_1) \neq 0$ then a similar reasoning as in the first part of the proof yields

$$\begin{aligned}|\operatorname{supp}\pi_i^R(\bar{D}_1 + \bar{D}_2 + \ldots + \bar{D}_h)| = |\operatorname{supp}\pi_i^R(V_{\alpha_1} \otimes w \oplus \overline{\rho_s^R(D_1)} + \bar{D}_2 + \ldots + \bar{D}_h)| = \\ \max\{|\operatorname{supp}\pi_i^R(V_{\alpha_1} \otimes w)|, |\operatorname{supp}\pi_i^R(\rho_s^R(\bar{D}_1) + \bar{D}_2 + \ldots + \bar{D}_h)|\}.\end{aligned}$$

First note that

$$\begin{array}{rcl}|\operatorname{supp}\pi_i^R(\overline{\rho_s^R(D_1)} + \bar{D}_2 + \ldots + \bar{D}_h)| & \leq & |\operatorname{supp}\pi_i^R(\rho_s^R(D_1) + D_2 + \ldots + D_h)| \\ & \leq & |\operatorname{supp}\pi_i^R(D_1 + \ldots + D_h)|\end{array}.$$

Secondly, since $\pi_i^R(D_1) \neq 0$, we also have

$$|\operatorname{supp}\pi_i^R(V_{\alpha_1} \otimes w)| \leq |\operatorname{supp}\pi_i^R(D_1)| \leq |\operatorname{supp}\pi_i^R(D_1 + \ldots + D_h)|\,.$$

Thus the claim follows.

q.e.d.

To extend the statement of Proposition 2 to products of t codes we need the following technical lemma.

Lemma 3. Let U be any code, not necessarily chained, and let $D \leq U \otimes W$ be a subspace of the form

$$D = U_1 \otimes w_1 \oplus U_2 \otimes w_2 \oplus \ldots \oplus U_l \otimes w_l$$

with $\{w_1, \ldots, w_l\}$ a basis of W and $U_i \leq U$. Assume that there are subspaces $\bar{U}_i \leq U$ with $\dim \bar{U}_i = \dim U_i$ and $\mid \operatorname{supp}(\bar{U}_{i_1} + \bar{U}_{i_2} + \ldots + \bar{U}_{i_p}) \mid \leq \mid \operatorname{supp}(U_{i_1} + U_{i_2} + \ldots + U_{i_p}) \mid$ for any $i_1, \ldots, i_p \in \{1, \ldots, l\}$. If

$$B(D) = \bar{U}_1 \otimes w_1 \oplus \bar{U}_2 \otimes w_2 \oplus \ldots \oplus \bar{U}_l \otimes w_l,$$

then $\dim B(D) = \dim D$ and $\mid \operatorname{supp} B(D) \mid \leq \mid \operatorname{supp} D \mid$. In particular, if $U = V$ is chained, we may take $\bar{U}_i = V_{\alpha_i}$ for $\alpha_i = \dim U_i$.

Proof. The assertion on the dimension is trivial. Furthermore we have

$$\begin{aligned} \mid \operatorname{supp} B(D) \mid &= \textstyle\sum_{j=1}^m \mid \operatorname{supp} \pi_j^R(B(D)) \mid = \sum_{j=1}^m \mid \operatorname{supp} \Sigma_{s=1}^l (w_s)_{(j)} \bar{U}_s \mid \leq \\ &\leq \textstyle\sum_{j=1}^m \mid \operatorname{supp} \Sigma_{s=1}^l (w_s)_{(j)} U_s \mid = \sum_{j=1}^m \mid \operatorname{supp} \pi_j^R(D) \mid = \mid \operatorname{supp} D \mid . \end{aligned}$$

In case when $U = V$ is chained it suffices to observe that the subspaces $\bar{U}_i = V_{\alpha_i}$ satisfy the assumptions.

q.e.d.

III. GENERAL PRODUCT CODES

From now on we consider $[n_i, k_i]$ codes V^i $(i \in \{1, \ldots, t\})$ and assume that all of them satisfy the chain condition. Let $\mathbf{V}^i = \{v_1^i, \ldots, v_{k_i}^i\}$ be a basis of V^i such that

$$\mid \operatorname{supp}\langle v_1^i, \ldots, v_r^i \rangle \mid = d_r(V^i) \quad (r \in \{1, \ldots, k_i\}).$$

Definition 4. Let $D \leq V^1 \otimes \ldots \otimes V^t$. We say that D is normal if it is generated by elements of $\mathbf{V}^1 \otimes \ldots \otimes \mathbf{V}^t = \{v_{i_1}^1 \otimes \ldots \otimes v_{i_t}^t \mid v_{i_j}^j \in \mathbf{V}^j \text{ for } j = 1, \ldots, t\}$ and it satisfies the following property:

Whenever $$v_{i_1}^1 \otimes v_{i_2}^2 \otimes \ldots \otimes v_{i_t}^t \in D$$

then $$v^1_{j_1} \otimes v^2_{j_2} \otimes \ldots \otimes v^t_{j_t} \in D$$

for all j_l with $1 \leq j_l \leq i_l$ $(l \in \{1, \ldots, t\})$.

Proposition 5. Let D be any subspace of $C = V^1 \otimes \ldots \otimes V^t$. Then there exists a normal $\bar{D} \leq C$ with

$$(i) \quad \dim D = \dim \bar{D} \text{ and}$$

$$(ii) \quad |\operatorname{supp} \bar{D}| \leq |\operatorname{supp} D|.$$

Proof. We first apply Proposition 2 to D with respect to $V = V^1 \otimes \ldots \otimes V^{t-1}$ and $W = V^t$. (Note that here and in the following the space W satisfies the chain condition, so the properties of V and W in Proposition 2 are changed.) We obtain $\bar{D} \leq C$ with $\dim \bar{D} = \dim D$ and $|\operatorname{supp} \bar{D}| \leq |\operatorname{supp} D|$. Moreover $\bar{D}$ is generated by elements of $\mathbf{B}^{t-1} \otimes \mathbf{V}^t$ for a suitable basis $\mathbf{B}^{t-1}$ of $V^1 \otimes \ldots \otimes V^{t-1}$. Next, we use both Proposition 2 and Lemma 3 to obtain inductively subspaces $B^s(D)$ for $s = t-1, t-2, \ldots, 1$ generated by elements of $\mathbf{B}^s \otimes \mathbf{V}^{s+1} \otimes \ldots \otimes \mathbf{V}^t$ for some subset $\mathbf{B}^s$ of $V^1 \otimes \ldots \otimes V^s$ such that $\dim B^s(D) = \dim D$ and $|\operatorname{supp} B^s(D)| \leq |\operatorname{supp} D|$. We proceed as follows:
Let $B^{t-1}(D) = \bar{D}$. If for $3 \leq s \leq t-1$ the space $B^s(D)$ has already been constructed then we may write $B^s(D) = U_1 \otimes x_1 \oplus \ldots \oplus U_{n_1 \ldots n_{s-1}} \otimes x_{n_1 \ldots n_{s-1}}$ where the elements x_i are in $\mathbf{V}^{s+1} \otimes \ldots \otimes \mathbf{V}^t$ and the subspaces U_i are contained in $V^1 \otimes \ldots \otimes V^s$. Let $\bar{U}_i$ denote a subspace obtained from U_i by applying Proposition 2 with $V = V^1 \otimes \ldots \otimes V^{s-1}$ and $W = V^s$. Thus $\bar{U}_i = U_{i1} \otimes y_1 \oplus \ldots \oplus U_{in_s} \otimes y_{n_s}$ with $U_{ij} \leq V^1 \otimes \ldots \otimes V^{s-1}$ and $y_j \in \mathbf{V}^s$. Now, we put $B^{s-1}(D) = \bigoplus_{i,j} U_{ij} \otimes y_j \otimes x_i$. By the second part of Proposition 2 and Lemma 3 we obtain $\dim B^{s-1}(D) = \dim B^s(D) = \dim D$ and $|\operatorname{supp} B^{s-1}(D)| \leq |\operatorname{supp} B^s(D)| \leq |\operatorname{supp} D|$. If we have constructed $B^2(D)$ then the subspaces U_i are in V^1 and to get $B^1(D)$ we can take as $\bar{U}_i$ the subspace of the chain of V^1 of the same dimension as U_i.
Note that the subspace $B^1(D)$ is generated by elements of $\mathbf{V}^1 \otimes \ldots \otimes \mathbf{V}^t$. Furthermore, $\dim B^1(D) = \dim D$ and $|\operatorname{supp} B^1(D)| \leq |\operatorname{supp} D|$.
Now, we will construct a normal subspace $N(D)$ of C with the properties as stated in the Proposition. Again, our construction is inductive. For each $s = 1, \ldots, t$ we define a subspace $N^s(D)$ with $\dim N^s(D) = \dim D$ and $|\operatorname{supp} N^s(D)| \leq |\operatorname{supp} D|$ as follows: We put $N^1(D) = B^1(D)$. If $N^{s-1}(D)$ has already been constructed we define $N^s(D)$ as the subspace obtained from $N^{s-1}(D)$ by applying the second part of Lemma 3 with $V = V^1 \otimes \ldots \otimes V^{s-1} \otimes V^{s+1} \otimes \ldots \otimes V^t$ and $W = V^s$.
Clearly, in each step we have $\dim N^s(D) = \dim N^{s-1}(D) = \dim D$ and $|\operatorname{supp} N^s(D)| \leq |\operatorname{supp} N^{s-1}(D)| \leq |\operatorname{supp} D|$. Let $N(D) = N^t(D)$. It remains to prove that $N(D)$ is

normal.

In order to see that we show by induction that for $s \in \{1, \dots, t\}$ the space $N^s(D)$ satisfies the following property:

Whenever $$v^1_{i_1} \otimes \dots \otimes v^s_{i_s} \otimes v^{s+1}_{i_{s+1}} \otimes \dots \otimes v^t_{i_t} \in N^s(D)$$

then $$v^1_{j_1} \otimes \dots \otimes v^s_{j_s} \otimes v^{s+1}_{i_{s+1}} \otimes \dots \otimes v^t_{i_t} \in N^s(D)$$

for all j_l with $1 \leq j_l \leq i_l \quad (l \in \{1, \dots, s\})$.

For $s = 1$ this is trivial, by Lemma 3. Assume now that the property holds for $s-1$, and that we have

$$v^1_{i_1} \otimes \dots \otimes v^s_{i_s} \otimes v^{s+1}_{i_{s+1}} \otimes \dots \otimes v^t_{i_t} \in N^s(D).$$

Next we put

$$U_{j_1,\dots,j_{s-1},i_{s+1},\dots,i_t} = \{v^s_h \mid h \in \{1, \dots, k_s\},\ v^1_{j_1} \otimes \dots \otimes v^{s-1}_{j_{s-1}} \otimes v^s_h \otimes v^{s+1}_{i_{s+1}} \otimes \dots \otimes v^t_{i_t} \in N^{s-1}(D)\}$$

for $1 \leq j_l \leq i_l$. By the induction hypothesis we have

$$U_{i_1,\dots,i_{s-1},i_{s+1},\dots,i_t} \subseteq U_{j_1,\dots,j_{s-1},i_{s+1},\dots,i_t}.$$

Now, looking at the definition of $N^s(D)$, we obtain

$$v^1_{j_1} \otimes \dots \otimes v^s_{j_s} \otimes v^{s+1}_{i_{s+1}} \otimes \dots \otimes v^t_{i_t} \in N^s(D)$$

for $1 \leq j_l \leq i_l \quad (l \in \{1, \dots, s\})$.

q.e.d.

Normal subspaces of $V^1 \otimes ... \otimes V^t$ are closely related to $(k_1, \dots, k_t)$-partitions.

If τ is a $(k_1, k_2, \dots, k_t)$-partition of r, we can associate to τ a normal subspace of $V^1 \otimes \dots \otimes V^t$ by setting

$$D = \langle v^1_{i_1} \otimes v^2_{i_2} \otimes \dots \otimes v^t_{i_t} \mid i_t \leq \tau((i_1, \dots, i_{t-1}))\rangle.$$

Note that $\dim D = r$. The fact that D is normal is a direct consequence of the definition of a $(k_1, k_2, \dots, k_t)$-partition.

Conversely, if D is a normal subspace of $V^1 \otimes \dots \otimes V^t$, we can define a $(k_1, k_2, \dots, k_t)$-partition τ of $r = \dim D$ by

$$\tau((i_1,\dots,i_{t-1})) = \mid \{v^t_{i_t} \mid v^1_{i_1}\otimes\dots\otimes v^t_{i_t}\in D\}\mid .$$

Proposition 6. Let D be a normal subspace of $V^1\otimes ...\otimes V^t$ and let τ be the associated $(k_1,k_2,\dots,k_t)$-partition of $r=\dim D$. Then

$$\mid \operatorname{supp} D\mid = \ \Sigma_{\underline{i}\in\mathcal{M}_t}\Pi_{j=1}^{t-1}(d_{i_j}(V^j)-d_{i_j-1}(V^j))d_{\tau(\underline{i})}(V^t).$$

Proof. We argue by induction on t. One easily checks the case $t=1$ (note that $\mathcal{M}_1=\{0\}$). Assume that the assertion holds true for products of $t-1$ codes and consider

$$\mathcal{N}_{t-1}=\{\underline{i}=(i_2,\dots,i_{t-1})\mid 1\le i_j\le k_j, 2\le j\le t-1\}.$$

For each $s\in\{1,\dots,k_1\}$ let $r_s=\Sigma_{\underline{i}\in\mathcal{N}_{t-1}}\tau((s,i_2,\dots,i_{t-1}))$ and

$$\begin{aligned} D_s &= \langle v^2_{i_2}\otimes\dots\otimes v^t_{i_t}\mid i_t\le\tau((s,i_2,\dots,i_{t-1}))\rangle \ = \\ &= \langle v^2_{i_2}\otimes\dots\otimes v^t_{i_t}\mid v^1_s\otimes v^2_{i_2}\otimes\dots\otimes v^t_{i_t}\in D\rangle\le V^2\otimes ...\otimes V^t. \end{aligned}$$

Obviously, D_s is normal and therefore

$$\mid \operatorname{supp} D_s\mid = \Sigma_{\underline{i}\in\mathcal{N}_{t-1}}\Pi_{j=2}^{t-1}(d_{i_j}(V^j)-d_{i_j-1}(V^j))d_{\tau((s,\underline{i}))}(V^t)$$

by the induction hypothesis. So we only have to prove that

$$\mid \operatorname{supp} D\mid = \Sigma_{s=1}^{k_1}(d_s(V^1)-d_{s-1}(V^1))\mid \operatorname{supp} D_s\mid .$$

But $D_1\ge D_2\ge\dots\ge D_{k_1}$ since D is normal. Therefore, the assertion is a direct consequence of the fact that V^1 satisfies the chain condition and $\{v^1_1,\dots,v^1_{k_1}\}$ is a basis for the chain. **q.e.d.**

As a consequence of Proposition 5 and Proposition 6 we obtain the main result

Theorem 7. If each of the codes $V^1,\dots,V^t$ satisfy the chain condition, then

$$d_r(V^1\otimes\dots\otimes V^t)=\min\{\nabla(\tau)\mid\tau \text{ a } (k_1,\dots,k_t)\text{-partition of } r\}.$$

For $t=2$ this contains the affirmative answer to the Wei-Yang conjecture. The methods here are completely different from the ones used by Schaathun and the main result has first been published in the preprint [7] (independently of [4]).

To conclude we demonstrate the methods which we developed above.

Example 8 Let K be a field with $|K| \geq 11$ and let ψ be a generator of the multiplicative group of K. Let V^i $(i = 1, 2, 3)$ be codes generated by the following matrices respectively,

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \psi & \psi^2 & \psi^3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \psi & \psi^2 & \psi^3 & \psi^4 & \psi^5 & \psi^6 & \psi^7 & \psi^8 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Denote by v_1^1, v_2^1 the rows of G_1 and by v_1^i, v_2^i, v_3^i the rows of G_i for $i = 2, 3$. We fix these vectors as basis for each of the corresponding codes. All three codes satisfy the chain condition and their weight hierarchies are, respectively,

$$\begin{aligned} (d_1^1, d_2^1) &= (1, 4) && \text{for } V^1 \\ (d_1^2, d_2^2, d_3^2) &= (1, 5, 7) && \text{for } V^2 \\ (d_1^3, d_2^3, d_3^3) &= (2, 11, 13) && \text{for } V^3. \end{aligned}$$

Moreover, the elements of our fixed basis generate the subspaces of a chain.

a) None of the codes $V^1 \otimes V^2$, $V^2 \otimes V^3$ and $V^1 \otimes V^3$ satisfies the chain condition. We will prove this for $V^2 \otimes V^3$; the other two cases are proved analogously. In the sequel we use the following obvious observations:
Let $x \in V^3$. If supp $x \not\subseteq \{1, 2\}$, then $|$ supp $x \cap \{3, 4, ..., 13\}$ $| \geq 9$ $\quad(\star)$
Let $y \in V^2$. If supp $x \not\subseteq \{1\}$, then $|$ supp $x \cap \{2, 3, ..., 7\}$ $| \geq 4$ $\quad(\star\star)$
Recall that we can regard the elements of $V^2 \otimes V^3$ as 13×7-matrices. Assume that we have a chain of subspaces $C_1 \leq C_2 \leq C_3$ of dimensions 1, 2 and 3 with minimum supports. In fact, using the Wei-Yang formula, it is very easy to determine the orders of these supports, which are 2, 10, 13; but we do not use that. Note that we have $C_1 = \langle v_1^2 \otimes v_1^3 \rangle$. Put $W = \langle v_1^2, v_2^2 \rangle \otimes v_1^3$. Observe that $|$ supp W $| = 10$ and that $W \leq V^2 \otimes v_1^3$. The latter subspace is precisely the set of elements v in $V^2 \otimes V^3$ such that supp $\pi_i^C(v) \subseteq \{1, 2\}$ for any $i \in \{1, ..., 7\}$. If for some $v \in C_2$ and some i as before we have supp $\pi_i^C(v) \not\subseteq \{1, 2\}$ then, by $(\star)$, there are at least 9 entries in the support of v not contained in the first two rows. This, together with the fact that $v_1^2 \otimes v_1^3 \in C_2$ implies that $|$ supp C_2 $| \geq 11$, which is a contradiction. Therefore $C_2 \leq V^2 \otimes v_1^3$ and from this it is easy to deduce that in fact $C_2 = W$.

So, the support of C_3 includes at least 10 entries in the first two rows. Now, let $w \in C_3 \setminus W$ and consider $\pi_j^R(w)$ for $j \in \{1, ..., 13\}$. If for some value of j, supp $\pi_j^R(w) \not\subseteq \{1\}$ then, by $(\star\star)$, $\mid$ supp $C_3 \mid \geq 14$. Otherwise $C_3 = V^2 \otimes v_1^3$ and $\mid \mathrm{supp} C_3 \mid = 14$. However, the subspace $v_1^2 \otimes V^3$ has support of order 13. This contradiction proves that $V^2 \otimes V^3$ does not satisfy the chain condition.

b) As a consequence of part a), the Wei-Yang formula can not be used to compute the weight hierarchy of the code $V^1 \otimes V^2 \otimes V^3$.

We are going to apply our results to compute the minimum support of a subspace of dimension 4 of $V_1 \otimes V_2 \otimes V_3$. To do that, first we compute the possible $(2,3,3)$-partitions τ of 4.

$$\begin{matrix} 3 & 0 \\ 1 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 2 & 0 \\ 2 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 2 & 2 \\ 0 & 0 \\ 0 & 0 \end{matrix}$$

$$\begin{matrix} 2 & 0 \\ 1 & 0 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 2 & 1 \\ 1 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 1 \\ 0 & 0. \end{matrix}$$

Here a block consists of the values of a partition τ where $\mathcal{M}_2$ is ordered as follows

$$\begin{matrix} (1,1) & (2,1) \\ (1,2) & (2,2) \\ (1,3) & (2,3). \end{matrix}$$

Each partition corresponds to a normal subspace of $V^1 \otimes V^2 \otimes V^3$. For example, the second partition of the first row corresponds to the normal subspace

$$A = \langle v_1^1 \otimes v_1^2 \otimes v_1^3, v_1^1 \otimes v_1^2 \otimes v_2^3, v_1^1 \otimes v_1^2 \otimes v_3^3, v_1^1 \otimes v_2^2 \otimes v_1^3 \rangle.$$

The support of this subspace can be computed by Proposition 6 as follows. The next table is obtained by multiplying the differences $d_1^1 = 1$, $d_2^1 - d_1^1 = 3$ with $d_1^2 = 1$, $d_2^2 - d_1^2 = 4$, $d_3^2 - d_2^2 = 2$, respectively.

$$\begin{matrix} 1 & 3 \\ 4 & 12 \\ 2 & 6. \end{matrix} \qquad (1)$$

Now, consider the table obtained from the partition by substituting each entry i by d_i^3, that is

$$\begin{array}{cc} d_3^3 & d_1^3 \\ d_0^3 & d_0^3 \\ d_0^3 & d_0^3 \end{array} = \begin{array}{cc} 13 & 2 \\ 0 & 0 \\ 0 & 0 \end{array} \qquad (2)$$

We multiply elementwise the tables (1) and (2).

$$\text{table(1)table(2)} = \begin{array}{cc} 13 & 6 \\ 0 & 0 \\ 0 & 0 \end{array}$$

The sum of the entries of the obtained table is $\mid$ supp $A \mid$. Hence $\mid$ supp $A \mid = 13 + 6 = 19$.

If we repeat this process with all the partitions, we get the orders of the supports of all the normal subspaces associated to our fixed basis.

$$\begin{array}{cccc} 21 & 19 & 55 & 44 \\ 23 & 25 & 20 & 40. \end{array}$$

Therefore, $d_4(V^1 \otimes V^2 \otimes V^3) = 19$ and the minimum support is attained by the subspace A.

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