The trivial intersection problem for characters of principal indecomposable modules

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Abstract For a prime p and a finite group G let $\Phi_p(G)$ denote the complex character associated to the projective indecomposable module in characteristc p with trivial head. Let $\mathrm{Irr}(\Phi_p(G))$ denote the set of irreducible characters occurring as constituents in $\Phi_p(G)$. We characterize all finite simple groups which satisfy $\mathrm{Irr}(\Phi_p(G)) \cap \mathrm{Irr}(\Phi_q(G)) = \{1_G\}$ for all primes $p \neq q$.

Keywords Finite simple group, irreducible character, principal indecomposable module, small rank permutation representation

1 Introduction

All groups in this note are assumed to be finite. By $\mathrm{B}_0(G)_p$ we denote the principal p-block for a prime p and by $\Phi_p(G)$ the complex character associated to the principal indecomposable module with respect to p, i.e., $\Phi_p(G)$ is the complex character afforded by the projective cover of the trivial module in characteristic p. Let $\mathrm{Irr}(\mathrm{B}_0(G)_p)$ be the set of all $\chi \in \mathrm{Irr}(G)$ belonging to $\mathrm{B}_0(G)_p$. Recently, Bessenrodt and Zhang proved

Theorem 1.1 [2] *A group G is nilpotent if and only if*

$$\mathrm{Irr}(\mathrm{B}_0(G)_p) \cap \mathrm{Irr}(\mathrm{B}_0(G)_q) = \{1_G\}$$

for all primes p and q with $p \neq q$.

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Now, let $\mathrm{Irr}(\Phi_p(G))$ denote the set of all $\chi \in \mathrm{Irr}(G)$ which occur as constituents in $\Phi_p(G)$. Since $\mathrm{Irr}(\Phi_p(G))$ is a subset of $\mathrm{Irr}(\mathrm{B}_0(G)_p)$ we may ask:

Which properties of G characterize the (weaker) condition

$$\mathrm{Irr}(\Phi_p(G)) \cap \mathrm{Irr}(\Phi_q(G)) = \{1_G\} \tag{1}$$

for all primes p, q with $p \neq q$?

The situation turns out to be more subtle as easy examples already show.

Lemma 1.2 *The group $G = \mathrm{PSL}(2, r)$, where r is a power of a prime, satisfies*

$$\mathrm{Irr}(\Phi_p(G)) \cap \mathrm{Irr}(\Phi_q(G)) = \{1_G\}$$

for all primes $p \neq q$ if and only if r is a Mersenne prime.

Proof: For a prime l we write shortly Φ_l for $\Phi_l(G)$. Using the decomposition matrices of $\mathrm{PSL}(2, r)$ which are given in [3] and Proposition 2.6 of the next section we see the following. If $r = 2^f$ then $\{1_G\} \subset \mathrm{Irr}(\Phi_2) \cap \mathrm{Irr}(\Phi_q)$ for all primes $q \mid r - 1$. If r is odd then $\{1_G\} \subset \mathrm{Irr}(\Phi_2) \cap \mathrm{Irr}(\Phi_q)$ for all primes $2 \neq q \mid r + 1$. Thus $\mathrm{Irr}(\Phi_2) \cap \mathrm{Irr}(\Phi_q) = \{1_G\}$ for all odd primes q implies that r is a Mersenne prime. Conversely, if r is a Mersenne prime then G has a 2-complement, namely a Borel subgroup. In this case Corollary 2.3 of the next section implies $\mathrm{Irr}(\Phi_p) \cap \mathrm{Irr}(\Phi_q) = \{1_G\}$ for all primes p, q with $p \neq q$. $\square$

The main result of this paper is the proof of

Theorem 1.3 *If G is a finite non-abelian simple group then the following conditions are equivalent.*

a) $\mathrm{Irr}(\Phi_p(G)) \cap \mathrm{Irr}(\Phi_q(G)) = \{1_G\}$ *for all primes p, q with $p \neq q$.*

b) $G = \mathrm{PSL}(2, r)$ *where r is a Mersenne prime.*

c) G has a 2-complement.

d) $\Phi_2(1) = |G|_2$.

The equivalence of b) and c) follows directly from ([9], Theorem 1). The equivalence of b) and d) has been proved in ([20], Theorem C). Note that the existence of a 2-complement implies the existence of a maximal subgroup whose index is a power of 2. This happens only for $\mathrm{PSL}(d, r)$ with $\frac{r^d-1}{r-1} = 2^a$, as shown by Guralnick in [9]. By Zsigmondy's Lemma (see 3.3), this number theoretical condition is satisfied if and only if $d = 2$ and r is a Mersenne prime. Thus $\mathrm{PSL}(2, r)$ with r a Mersenne prime is the only non-abelian simple group which has a 2-complement.

Note, that for any finite group G the 2-part of the degree of $\Phi_2(G)$ is exactly the 2-part $|G|_2$ of the order of G (see [8]).

We would like to mention here that for a simple group the condition

$$\mathrm{Irr}(\Phi_2(G)) \cap \mathrm{Irr}(\Phi_p(G)) = \{1_G\} \quad \text{for all primes } p \neq 2$$

does not imply that G has a 2-complement, as $\mathrm{PSL}(3,7)$ shows. This is in contrast to a local version of the Bessenrodt-Zhang Theorem which follows by a slight refinement of their proof, and says:

If p is a fixed prime then

$$\mathrm{Irr}(\mathrm{B}_0(G)_p) \cap \mathrm{Irr}(\mathrm{B}_0(G)_q) = \{1_G\} \quad \textit{for all primes } q \neq p$$

implies that G is p-nilpotent.

If we drop the assumption *simple group* in Theorem 1.3 we do not know a characterization of groups G which satisfy $\mathrm{Irr}(\Phi_p(G)) \cap \mathrm{Irr}(\Phi_q(G)) = \{1_G\}$ for all primes p, q with $p \neq q$. For instance, this condition is achieved by the group $\mathrm{PSL}(3,7) \cdot 2$.

2 General results

Let $(\cdot,\cdot)$ denote the Frobenius product of complex characters, i.e. the usual bilinear form in the ring of characters. To be brief we write Φ_p for $\Phi_p(G)$ if there is no confusion with the group in question. Since the trivial character 1_G appears always with multiplicity one in Φ_p for all p we have $(\Phi_p, \Phi_q) \geq 1$ for all primes p, q. Thus the condition

$$\mathrm{Irr}(\Phi_p) \cap \mathrm{Irr}(\Phi_q) = \{1_G\}$$

is equivalent to

$$(\Phi_p, \Phi_q) = 1.$$

Definition 2.1 We say that G is a TI-*group for characters* if $(\Phi_p, \Phi_q) = 1$ for all primes $p \neq q$.

Proposition 2.2 *Let G be a finite group and let p, q be different primes. Suppose that $G = HL$ with $H, L \leq G$ where $p \nmid |H|$ and $q \nmid |L|$. Then*

$$(1_H^G, 1_L^G) = 1.$$

In particular, $(\Phi_p, \Phi_q) = 1$.

Proof: By Frobenius reciprocity and Mackey's Theorem, we have

$$(1_H^G, 1_L^G) = (1_H, (1_L^G)_H) = (1_H, 1_{L\cap H}^H) = (1_{L\cap H}, 1_{L\cap H}) = 1.$$

For the last statement note that Φ_p is a constituent of 1_H^G and Φ_q is of 1_L^G (see for example [21] Theorem 2.27).

□

The following Corollaries are immediate consequences of Proposition 2.2.

Corollary 2.3 *Let p be a prime. If G has a p-complement then*

$$(\Phi_p, \Phi_q) = 1$$

for all primes q with $q \neq p$.

Proof: Apply Proposition 2.2 with L a Sylow p-subgroup and H a p-complement. $\square$

Corollary 2.4 *If for all primes p with possibly one exception the group G has a p-complement then G is a* TI*-group for characters. In particular, solvable groups are* TI*-groups for characters.*

Example 2.5 Let $G = \mathrm{A}_5$ be the alternating group on 5 letters. The irreducible characters of G are $1_G, \chi_3, \chi_{3^*}, \chi_4, \chi_5$ with degrees $1, 3, 3, 4, 5$. One easily sees that

$$\begin{array}{rcl} \Phi_2 &=& 1_G + \chi_3 + \chi_{3^*} + \chi_5 \\ \Phi_3 &=& 1_G + \chi_5 \\ \Phi_5 &=& 1_G + \chi_4. \end{array}$$

Hence

$$(\Phi_2, \Phi_3) = 2,\ (\Phi_2.\Phi_5) = 1 \text{ and } (\Phi_3, \Phi_5) = 1.$$

Observe that G has a 5-complement, but neither a 2- nor a 3-complement.

Proposition 2.6 *Let q be an odd prime and suppose that there exists a character χ of odd degree with $1_G \neq \chi = \overline{\chi} \in Irr(\Phi_q)$. Then*

$$(\Phi_2, \Phi_q) > 1.$$

Proof: On $2'$-elements we have

$$\chi = \sum_{\beta \in \mathrm{IBr}_2(G)} a_\beta \beta$$

where $\mathrm{IBr}_2(G)$ denotes the set of irreducible 2-Brauer characters. Clearly, the trivial character is the only real-valued linear 2-Brauer character. Since $\chi = \bar{\chi}$ we obtain

$$\chi = a 1_G + \sum_{\beta \neq \bar{\beta}} a_\beta (\beta + \overline{\beta}) + \sum_{1_G \neq \beta = \overline{\beta}} a_\beta \beta.$$

Fong's Lemma (see [14], Vol. II, Theorem 8.13) yields $2 \mid \beta(1)$ for all β with $1_G \neq \beta = \overline{\beta}$. Since $\chi(1)$ is odd we get $a \geq 1$. This shows that the trivial character is a constituent of the modulo 2 reduction of χ which means that χ is a constituent of Φ_2. $\square$

Corollary 2.7 *Let q be an odd prime. If $2 \mid \Phi_q(1)$ then $(\Phi_2, \Phi_q) > 1$.*

Proof: Note that Φ_q is real-valued. Thus we may write

$$\Phi_q = 1_G + \sum_{\chi \neq \bar{\chi}} a_\chi (\chi + \overline{\chi}) + \sum_{1_G \neq \chi = \overline{\chi}} a_\chi \chi \quad \text{with } a_\chi \in \mathrm{N}$$

where $\chi \in \mathrm{Irr}(G)$. Since $2 \mid \Phi_q(1)$ there exists an irreducible character χ of odd degree with $1_G \neq \chi = \overline{\chi}$ and an application of Proposition 2.6 completes the proof. $\square$

For general groups we do not know much about the degree $\Phi_p(1)$ apart from Dixon's result that $|G|_p$ divides $\Phi_p(1)$. As a warning we would like to mention that even for simple groups it can happen that $\Phi_p(1)$ is even only for the prime $p = 2$. For instance, the Mathieu group M_{11} serves as an example.

Proposition 2.8 *Let G be a finite group with a transitive permutation action of degree $n > 1$. Suppose that the prime factorization of n contains t different primes. If the complex character of the action contains $k \leq t$ different irreducible characters then*

$$(\Phi_p, \Phi_q) > 1$$

for suitable primes $p \neq q$ with $pq \mid n$.

Proof: If π denotes the complex character of the permutation action of G then

$$\pi = 1_G + \sum_{i=1}^{s-1} e_i \chi_i$$

where $e_i \in \mathrm{N}$ and $1_G \neq \chi_i \in \mathrm{Irr}(G)$. Let p be a prime with $p \mid n$. Furthermore, let M be the permutation module over a field K of characteristic p with permutation basis $v_1, \ldots, v_n$. Then M carries a non-degenerate G-invariant bilinear form b defined by

$$(v_i, v_j) = \delta_{ij}.$$

Since $p \mid n$ the form b is 0 on the trivial submodule $S = K(v_1 + \ldots + v_n)$. Thus M contains a second trivial module as a composition factor, namely $M/S^{\perp} \cong S^* \cong S$. This means that the modulo p reduction of at least one χ_i contains the trivial module. The condition $t > s - 1$ implies $(\Phi_p, \Phi_q) > 1$ for some primes $q \neq p$ with $p, q \mid n$. □

Corollary 2.9 *Let G be a doubly transitive group of degree n where n is not the power of a prime. Then for all primes $p \neq q$ with $pq \mid n$ we have*

$$(\Phi_p, \Phi_q) > 1.$$

Proof: The permutation character π of a doubly transitve action satisfies $\pi = 1_G + \chi$ with $1_G \neq \chi \in \mathrm{Irr}(G)$. Thus by the proof of Proposition 2.8 we get the assertion. □

Corollary 2.10 *If G is a solvable transitive group of degree n and the prime factorization of n contains t different primes then the corresponding complex permutation character contains at least $t + 1$ different irreducible characters.*

Proof: This follows immediately from Proposition 2.8 and Corollary 2.4. □

Recall that a finite group G has a *rank k action of degree n* if G acts transitively on a set Ω with $|\Omega| = n$ and the stabilizer of a point has k orbits on Ω. Let

$$\pi = \sum_{i=1}^{s} e_i \chi_i = 1_G + \sum_{i=2}^{s} e_i \chi_i$$

with $e_i \in \mathrm{N}$ and pairwise different $\chi_i \in \mathrm{Irr}(G)$ denote the complex permutation character of a rank k action. Then, by ([28], 29.2),

$$1 + \sum_{i=2}^{s} e_i^2 = k.$$

In particular, π has at most k different irreducible constituents and exactly k if $k \leq 4$.

3 Chevalley groups

In this section we prove

Proposition 3.1 *A (simple) Chevalley group G is a* TI*-group for characters if and only if $G = \mathrm{PSL}(2, r)$ where r is a Mersenne prime.*

The first two results are well-known.

Lemma 3.2 *Let p and r be primes. If*

$$r^n + 1 = p^m$$

for $n, m \in \mathrm{N}$ then one of the following cases occur.

(i) $2^3 + 1 = 3^2$.

(ii) $m = 1, r = 2$ *and p is a Fermat prime.*

(iii) $n = 1, p = 2$ *and r is a Mersenne prime.*

Lemma 3.3 (Zsigmondy, [14], chap. IX, 8.3) *Let r be a power of a prime and $2 \leq d \in \mathrm{N}$. Then there exists a prime p with*

(1) $p \mid r^d - 1$.

(2) $p \nmid r^i - 1$ *for* $0 < i < d$.

unless $d = 6, r = 2$ or $d = 2, r = 2^b - 1$. We call such a prime p a primitive prime divisor of $r^d - 1$.

In the exceptional case $d = 6$ and $r = 2$ we still can find for any $0 < i < 6$ a prime p dividing $2^6 - 1$ but not dividing $2^i - 1$. Note that in this case the prime p depends on i.

Definition 3.4 We say that a prime power r is *exceptional* if $r = 8$ or r is a Mersenne prime or $r + 1$ is a Fermat prime. Thus, if r is not exceptional then there are primes $p \neq q$ with $pq \mid r + 1$.

Proposition 3.5 *Let $G(r)$ be a non-twisted Chevalley group defined over the finite field $\mathbb{F}_r$. If r is not exceptional then*

$$(\Phi_p, \Phi_q) > 1$$

for all primes $p \neq q$ with $pq \mid r + 1$.

Proof: This follows immediately from Theorem B of Hiss [11]. □

Corollary 3.6 *Let $G(r)$ be a non-twisted Chevalley group with r odd. If $G(r)$ is a* TI*-group then r is a Mersenne prime.*

In order to prove Proposition 3.1 we first rule out the cases where the groups have a doubly transitive permutation representation. These groups are determined in [4].

Lemma 3.7 *The groups* $\mathrm{PSU}(3,r)$, ${}^2B_2(2^{2m+1})$ *and* ${}^2G_2(3^{2m+1})$ *are not* TI*-groups for characters.*

Proof: $\mathrm{PSU}(3,r)$ has a doubly transitive action of degree r^3+1. By Lemma 3.2, r^3+1 is not a prime power since r is a square (hence $r \neq 2$) and r^3+1 can not be a Fermat prime. Thus the assertion follows by Corollary 2.9.

The Suzuki groups ${}^2B_2(2^{2m+1})$ are doubly transitive of degree r^2+1 for $r = 2^{2m+1}$ and we may apply again Corollary 2.9. Note that for $r = 8$ the degree $r^2+1 = 65$ is not a power of a prime.

The Ree groups ${}^2G_2(3^{2m+1})$ have a doubly transitive action of degree r^3+1 where $r = 3^{2m+1}$. Since r^3+1 is not a power of a prime we may apply again Corollary 2.9. $\square$

In order to deal with projective special linear groups we need a number theoretical fact.

Lemma 3.8 *Let* r *be a power of a prime and* $2 < d \in \mathrm{N}$ *with* $(d,r) \neq (3,8)$*. Assume that*

(i) either d *is not a prime,*

(ii) or $r = 8$*,*

(iii) or $r = 2^{2^t}$ *with* $t \geq 1$*.*

Then

$$n = \frac{r^d - 1}{r - 1}$$

contains at least 2 *different primes.*

Proof: If d is not a prime then we can choose an $s \mid d$ with $1 < s < d$. Let p be a primitive prime divisor of $r^d - 1$ and let q be any prime divisor of $\frac{r^s-1}{r-1}$. Then $p \neq q$ and $pq \mid n$.

If $r = 8 = 2^3$ then $d \neq 2, 3$ and

$$\frac{r^d - 1}{r - 1} = \frac{2^{3d} - 1}{2^3 - 1}.$$

As $3d \neq 6$ we may choose a primitive prime divisor p of $2^{3d} - 1$, and as $d > 3$ we may choose a further prime $q \mid 2^d - 1$ with $q \nmid 2^3 - 1$. Obviously, $p \neq q$ and $p, q \mid 2^{3d} - 1$. Moreover, $p, q \nmid 2^3 - 1$. Hence $p, q \mid n$.

In the case (iii) we have $r = 2^{2^t}$ with $t \geq 1$, hence

$$n = \frac{2^{2^t d} - 1}{2^{2^t} - 1}.$$

If $2^t d = 6$ then either $(d,r) = (3,4)$ or $(d,r) = (6,2)$. In the first case $n = 21$ and in the second $n = 63$. So we may assume $6 \neq 2^t d$ and we can choose a primitive prime divisor p of $2^{2^t d} - 1$. Furthermore, as $2^t < 2^{t-1} d$, we may also choose a prime q with $q \mid 2^{2^{t-1} d} - 1$ but $q \nmid 2^{2^t} - 1$. By construction $p, q \mid n$ and $p \neq q$. $\square$

Lemma 3.9 *For* $d \geq 4$ *the group* $G = \mathrm{PSL}(d,r)$ *has a rank* 3 *permutation representation on the set of* 2*-dimensional subspaces of the natural vector space* $V = V(d,r)$*.*

Proof: Clearly, G acts transitively on the set $V^{(2)}$ of 2-dimensional subspaces of V. If $H_0 \in V^{(2)}$ then one checks straightforward that the stabilizer G_{H_0} of H_0 in G has the following three orbits

$$H_0, \quad \{H \mid H \in V^{(2)},\ H \cap H_0 = \{0\}\}, \quad \{H \mid H \in V^{(2)},\ \dim H \cap H_0 = 1\}.$$

Thus G has a rank 3 action on $V^{(2)}$. □

Lemma 3.10 *Let $d \geq 3$. Then $G = \mathrm{PSL}(d,r)$ is a TI-group for characters if and only if $G = \mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$.*

Proof: Suppose that $(\Phi_p, \Phi_q) = 1$ for all primes $p \neq q$. Note first that, by Proposition 3.5, we may assume that r is exceptional.
a) We claim that d is a prime and either $r = 2$ or is a Mersenne prime.
If $G = \mathrm{PSL}(3,8)$ then $(\Phi_2, \Phi_7) > 1$, by the GAP library (see Table 1). Thus we may assume that $(d,r) \neq (3,8)$. The group $\mathrm{PSL}(d,r)$ has a doubly transitive action of degree $n = \frac{r^d-1}{r-1}$ on the lines of the natural vector space $V = V(d,r)$ (see [4]). In addition, by Corollary 2.9, we may assume that n is a prime power. Thus, by Lemma 3.8, we get the claim.
b) Let $d = 3$ and note that $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ which we considered already in Lemma 1.2. If $r \neq 2$ then, by part a), $G = \mathrm{PSL}(3,r)$ where r is a Mersenne prime and $n = 1+r+r^2$ is a power of a prime. Thus $r = 2^t - 1$ for a prime t. In case $r = 3$ we have $G = \mathrm{PSL}(3,3)$ which can be checked directly with the GAP library (see Table 1). Thus t is odd and we get $2^t \equiv -1 \bmod 3$, hence $r \equiv 1 \bmod 3$, and therefore

$$n = 1 + r + r^2 \equiv 0 \bmod 3.$$

On the other hand, modulo 9 we obtain three possibilities for 2^t, namely

$$2^t \equiv 2, -1 \text{ or } 5 \bmod 9,$$

hence $r \equiv 1, -2$ or $4 \bmod 9$, and therefore in each case

$$n = 1 + r + r^2 \equiv 3 \bmod 9.$$

This shows that $3 \mid n$, but $9 \nmid n$. Thus $n = 1 + r + r^2 = 3$, a contradiction to $r > 1$.
c) Finally, let $d \geq 4$, hence by part a) $d \geq 5$ and $d - 1 = 2a$. The space $V(d,r)$ contains exactly

$$m = \Big(\frac{r^d-1}{r-1}\Big)\Big(\frac{r^{d-1}-1}{r^2-1}\Big)$$

2-dimensional subspaces on which G allows a rank 3 permutation representation, by Lemma 3.9. Furthermore, by Proposition 2.8, it is enough to show that m contains at least 3 different primes. If $(d,r) = (7,2)$ then $m = 43 \cdot 7 \cdot 3^2$ and the claim follows. In all the remaining cases we may choose a primitive prime divisor p_1 of $r^d - 1$ and a primitive prime divisor p_2 of $r^{d-1} - 1$. If moreover $a \geq 3$ then take $p_3 \mid r^a - 1$ with $p_3 \nmid r^2 - 1$. Then the three primes p_1, p_2 and p_3 are different and $p_i \mid m$ for $i = 1,2,3$. Assume finally that $a = 2$, hence $d = 5$. In particular

$$m = \frac{r^5-1}{r-1}(r^2+1).$$

If r^2+1 contains at least two different primes we are done again. In case r^2+1 is a prime power we get that r is even. By part a), we obtain $r=2$. But $G=\mathrm{PSL}(5,2)$ is not a TI-group, by the GAP library (see Table 1). $\square$

The classical symplectic, orthogonal and unitary groups allow a rank 3 permutation action on the set of isotropic lines of the underlying vector space (see [18], Theorem 1.1).

Lemma 3.11 *Let r be a power of a prime and $2 < d \in \mathrm{N}$ with $(d,r) \neq (3,2)$. Then $n = \frac{r^{2d}-1}{r-1}$ contains at least 3 different primes.*

Proof: As $(d,r) \neq (3,2)$ we may choose a primitive prime divisor p of $r^{2d}-1$ and a prime divisor q of r^d-1 with $q \nmid r^2-1$. If t is any prime divisor of $r+1$ then p,q,t are pairwise different and $p,q,t \mid n$. $\square$

Lemma 3.12 *The group G is not a TI-group for characters if G is one of the following groups.*

a) $\mathrm{PSp}(2d,r)$ *with $d>2$.*

b) $\Omega(2d+1,r)$ *with $d>2$.*

Proof: First note that $G=\mathrm{PSp}(6,2) \cong \Omega(7,2)$. For this group we have $(\Phi_2,\Phi_3)>1$, by the GAP library (see Table 1). Thus we may assume that $(d,r) \neq (3,2)$. The rank 3 permutation representation for both groups $\mathrm{PSp}(2d,r)$ and $\Omega(2d+1,r)$ is of degree $n=\frac{r^{2d}-1}{r-1}$. For the symplectic group this is obvious, for the orthogonal group see for instance ([13], Proposition 9.18). By Lemma 3.11, the degree n contains at least 3 different primes and Proposition 2.8 applies. $\square$

Lemma 3.12 does not cover the case $d=2$, hence $G=\mathrm{PSp}(4,r)$. For this group we have $n=(r^2+1)(r+1)$. By Proposition 2.8, we may assume that r^2+1 and $r+1$ are prime powers. In particular, $r=2^f$. Furthermore $\mathrm{PSp}(4,r)=\Omega(5,r)$ for r even and $\mathrm{PSp}(4,2) \cong \mathrm{S}_6$.

Lemma 3.13 *For $\mathrm{PSp}(4,r)$ with $r=2^f>2$ we have $(\Phi_2,\Phi_p)>1$ for all primes p with $p \mid r+1$.*

Proof: Let $G=\mathrm{Sp}(4,r)=\mathrm{PSp}(4,r)$ with $r=2^f>2$. By [29], section 2, Table II, the irreducible character χ_3 satisfies $(\Phi_p,\chi_3)=1$ for all primes $p \mid r+1$. On the other hand, χ_3 is denoted by ψ_2 in [1], and it is proved in Theorem 3 of [1] that $(\Phi_2,\psi_2)=\frac{r}{2}$. Thus we have $(\Phi_2,\Phi_p)>1$ for all primes $p \mid r+1$. $\square$

Lemma 3.14 *Let $4 \leq d \in \mathrm{N}$ and*

$$n=(r^{d-1}+1)\frac{r^d-1}{r-1}.$$

Then n contains at least three different primes unless $r=2$, $d=4$ or $r=2$ and $2^{d-1}+1$, 2^d-1 are primes. In the latter case d is also a prime.

Proof: First assume that $r^{d-1}+1$ is not a power of a prime. Thus we may choose primes $p_1 \neq p_2$ with $p_1, p_2 \mid r^{d-1}+1$. Observe that

$$\gcd(r^{d-1}+1, r^d-1) \mid r+1 \mid r^2-1.$$

Therefore, if we choose $q \mid r^d-1$ with $q \nmid r^2-1$ (recall that $d>2$) then $q \mid n$ and $p_1, p_2 \neq q$. Thus we have found three different primes dividing n.

So we are left with the case when $r^{d-1}+1$ is a power of a prime. Lemma 3.2 (note that $d \geq 4$) implies that either $r=2$, $d=4$ or $p=r^{d-1}+1$ is a Fermat prime. In particular $r=2^{2^t}$. If $t \geq 1$ we may apply Lemma 3.8 and we get two different primes $q_1, q_2 \mid \frac{r^d-1}{r-1}$. Furthermore, if $r=2$ it is also possible to find two different primes $q_1, q_2 \mid 2^d-1$ unless 2^d-1 is a Mersenne prime. In both cases $q_1, q_2 \neq p$ since $p \nmid r+1$. □

Lemma 3.15 *$G = \mathrm{P}\Omega^+(2d, r)$ $(d \geq 4)$ is not a* TI*-group for characters.*

Proof: By ([13], Proposition 9.18), there are

$$\frac{r^{2d-1}+r^d-r^{d-1}-1}{r-1} = (r^{d-1}+1)\frac{r^d-1}{r-1}$$

isotropic lines on which G allows a rank 3 permutation representation (see [18]). Thus the result follows by Proposition 2.8 and Lemma 3.14 unless $r=2$, $d=4$ or $r=2$ and both $2^{d-1}+1$ and 2^d-1 are primes. In particular, d is also a prime. The group $\mathrm{P}\Omega^+(8,2)$ can be checked directly by the GAP library (see Table 1). Thus we have to deal with the second case. For $d=5$ the assertion follows again by the GAP library (see Table 1). Therefore we may assume that $d>5$. The group $G = \Omega^+(2d,2) = \mathrm{P}\Omega^+(2d,2)$ acts transitively on the set of anisotropic subspaces of dimension 2 according to Witt's Theorem. By ([6], Lemma 2.1 (ii)), the number of such subspaces is

$$n = \frac{2^{2(d-1)}(2^d-1)(2^{d-1}-1)}{6} = 2^{2d-3}(2^d-1)\frac{2^{d-1}-1}{3}.$$

Furthermore, by ([6], Theorem 6.5 (ii)), the rank of the permutation action is $\frac{2^2+4+2}{2} = 5$. We claim that n contains at least five different primes which completes the proof by Proposition 2.8. Since $2^{d-1}+1$ is a Fermat prime we have $d-1=2^s$ for some s. Furthermore $d=2^s+1$ must be a prime. The condition $d>5$ implies $s \geq 4$. Now we choose a primitive prime divisor p_1 of $2^{d-1}-1 = 2^{2^s}-1$, a primitive prime divisor p_2 of $2^{2^{s-1}}-1$ and a primitive prime divisor p_3 of $2^{2^{s-2}}-1$. The choice of these primes implies that p_1, p_2, p_3 are pairwise distinct and $p_i \nmid 2^2-1=3$ for $i=1,2,3$. Thus

$$p_1p_2p_3 \mid \frac{2^{d-1}-1}{3}.$$

Finally, if we put $p_4 = 2^d-1$ and $p_5=2$ we have found five different primes dividing n. □

Lemma 3.16 *Let $4 \leq d \in \mathrm{N}$ and*

$$n = (r^d+1)\frac{r^{d-1}-1}{r-1}.$$

Then n contains at least three different primes unless $r=2$ and 2^d+1, $2^{d-1}-1$ are primes. In the latter case $d-1$ is also a prime.

Proof: We argue similary as in the proof of Lemma 3.14. □

Lemma 3.17 *$G = \mathrm{P}\Omega^-(2d, r)$ ($d \geq 4$) is not a* TI*-group for characters except possibly in case $r = 2$, $2^d + 1$ is a Fermat prime and $2^{d-1} - 1$ is a Mersenne prime.*

Proof: By ([13], 9.18), the number of isotropic lines is

$$\frac{r^{2d-1} - r^d + r^{d-1} - 1}{r - 1} = (r^d + 1)\frac{r^{d-1} - 1}{r - 1}.$$

Since the action of G on these lines is of rank 3 (see [18]) the result follows by Proposition 2.8 and Lemma 3.16. □

The argument in Lemma 3.17 does not apply for $\mathrm{P}\Omega^-(8, 2)$ and $\mathrm{P}\Omega^-(16, 2)$ since we have the prime factorizations $(2^4 + 1)(2^3 - 1) = 17 \cdot 7$ and $(2^8 + 1)(2^7 - 1) = 257 \cdot 127$.

In order to deal with the cases we left open in Lemma 3.17 we need

Lemma 3.18 *If $d \geq 10$ and $2^d + 1$ is a Fermat prime then*

$$f(d) = \frac{(2^{2(d-1)} - 1)(2^d + 1)(2^{d-2} - 1)}{3}$$

contains at least 6 different primes.

Proof: Let $d = 2a$ with $a = 2^f \geq 5$. We may choose primitive prime divisors $p_1, \ldots, p_4$ of

$$2^{2(d-1)} - 1,\ 2^{d-1} - 1,\ 2^{d-2} - 1,\ 2^{a-1} - 1.$$

Obviously, the primes $p_1 \ldots, p_4$ are pairwise different and $p_i \mid f(d)$ for all i. Furthermore, $p_i \neq 3$ for all i as $a \geq 5$. If we put $p_5 = 2^d + 1$ and $p_6 = 3$ then $p_1, \ldots, p_6$ are pairwise distinct and $p_i \mid f(d)$ for $i = 1, \ldots, 5$. For $p_6 = 3$ note that $3 \mid 2^{2(d-1)} - 1$ and $3 \mid 2^{d-2} - 1$ since d is even. Hence $3 \mid f(d)$. □

Lemma 3.19 (Dempwolff) *Let $G = \mathrm{PO}^-(2d, 2)$ and let k denote the rank of the action of G on the set of 2-dimensional totally isotropic subspaces of the underlying orthogonal vector space. Then*

a) $k = 5$ if $d = 4$.

b) $k = 6$ if $d \geq 5$.

Proof: Let $A \neq B$ be two totally isotropic subspaces of dimension 2. By Q we denote the quadratic form and by b the associated symplectic form. We put $W = A + B$ and claim that W has one of the following isometry types.

(i) dim $W = 3$ and W totally isotropic.

(ii) dim $W = 3$ and $W = (A \cap B) \perp H$ where H is a hyperbolic plane.

(iii) dim $W = 4$ and W not degenerate of type $\mathrm{O}^+(4, 2)$.

(iv) $\dim W = 4$ and W totally isotropic.

(v) $\dim W = 4$ and $W = R \perp H$ where H is a hyperbolic plane and R totally isotropic.

Let R denote the radical of W with respect to the symplectic form b, hence

$$R = \mathrm{rad}(W) = \{w \in W \mid b(w,x) = 0, \text{ for all } x \in W\}.$$

First we consider the case $\dim A \cap B = 1$, hence $\dim W = 3$.

Suppose that W is not totally isotropic. Thus $Q(w) \neq 0$ for some $w \in W$. If $R = W$ then $W = U \perp \langle w \rangle$ where U is totally isotropic. Thus U is the set of isotropic vectors in W contradicting $A, B \subseteq W$. It follows $R < W$. Thus $\dim R = 1$ and $R = A \cap B$. Moreover, since W/R is a non-degenerate symplectic space via the lift of b we get $W = (A \cap B) \perp H$ where H is a hyperbolic plane. Note that H can not be anisotropic as W contains 2-dimensional totally isotropic subspaces.

Now let $A \cap B = 0$, hence $\dim W = 4$.

If W is non-degenerate then W is of type $\mathrm{O}^+(4,2)$ since $\mathrm{O}^-(4,2)$ does not have a totally isotropic 2-dimensional subspace. Suppose that W is degenerate but not totally isotropic. Thus there is a $w \in W$ with $Q(w) \neq 0$ and $\dim R = 2$. The space W/R is non-degenerate and symplectic with respect to the lift of b. Thus $W = R \perp U$ where U is non-degenerate. If R is totally isotropic then $U = H$ is a hyperbolic plane.

Finally, we show that R must be totally isotropic. Suppose that R is not totally isotropic. Clearly, R contains isotropic vectors and we may assume that $R_0 = \langle w_0 \rangle$ is the set of isotropic vectors in R. Obviously, $R < A + R$ since A is totally isotropic. Furthermore, $A + R < W$ since otherwise $R < \mathrm{rad}(W)$. This shows that $A \cap R \neq 0 \neq B \cap R$. It follows $R_0 \subset A \cap B = 0$, a contradiction.

It is easy to see that the orthogonal group $\mathrm{O}(W)$ acts transitively on the set of pairs of totally isotropic subspaces of dimension 2 which generate W. Thus an application of Witt's Theorem completes the proof, observing that for $d = 4$ there are no totally isotropic subspaces of dimension 4. □

Lemma 3.20 *The group $G = \mathrm{P}\Omega^-(2d, r)$ $(d \geq 4)$ is not a* TI*-group for characters except of possibly $G = \mathrm{P}\Omega^-(16, 2)$.*

Proof: By Lemma 3.17, we may assume that $r = 2$ and $2^d + 1$ is a Fermat prime. Thus $d = 2^f$ with $f \geq 2$. For $d = 4$ the assertion follows by the GAP library (see Table 1). Thus we may suppose that $d \geq 16$. By Lemma 2.1 of [6], the number of isotropic lines in the underlying orthogonal vector space is

$$f(d) = \frac{(2^{2(d-1)} - 1)(2^d + 1)(2^{d-2} - 1)}{3}.$$

The previous Lemma 3.19 shows that the action of $\mathrm{PO}^-(2d, 2)$ on the set of 2-dimensional totally isotropic subspaces is of rank 6, and by Lemma 3.18, $f(d)$ has at least 6 different prime divisors. Thus, by Proposition 2.8, there are primes $p \neq q$ such that $(\Phi_p, \Phi_q) > 1$ for $\mathrm{PO}^-(2d, 2)$. The assertion then also holds for $\mathrm{P}\Omega^-(2d, 2)$ because of the following observation. □

Lemma 3.21 *Let $G \trianglelefteq H$ with H/G solvable. If $1 < (\Phi_p(H), \Phi_q(H))$ then the same holds true for G.*

Proof: Let $1 \neq \chi \in \mathrm{Irr}\Phi_p(H) \cap \mathrm{Irr}\Phi_q(H)$. Then both the reductions of χ modulo p and q contain the corresponding trivial Brauer character. Therefore if $G \leq \mathrm{Ker}\chi$ the same happens with respect to the group H/G so we get $1 < (\Phi_p(H/G), \Phi_Q(H/G)))$ which contradicts 2.4. Finally the fact that $\Phi_p(H)_G = e\Phi_p(G)$ for some $e \in \mathrm{N}$ implies that some irreducible $1_G \neq \phi \leq \chi$ is a constituent in both $\Phi_p(G)$, $\Phi_q(G)$. □

The next Lemma finishes the proof for orthogonal groups.

Lemma 3.22 $G = \mathrm{P}\Omega^-(16, 2)$ *is not a* TI-*group for characters.*

Proof: By ([6], Lemma 2.1), the underlying orthogonal vector space for G contains

$$n = (2^7 - 1)(2^8 + 1) = 127 \cdot 257$$

isotropic lines. The permutation action of G on these lines is of rank 3 (see [18]). If π denotes the associated complex permutation character then

$$\pi = 1 + \chi + \psi$$

with non-trivial irreducible characters $\chi \neq \psi$. Let M denote the permutation module over $\mathbb{F}_2$. Using the computer algebra sytem MAGMA, E. O'Brien proved that M contains the trivial module 7 times as a composition factor. Thus χ or ψ (maybe both) contain the trivial character if reduced modulo 2. Moreover, by the proof of Proposition 2.8, we may assume that w.l.o.g. the reduction of χ modulo 127 and the reduction of ψ modulo 257 contain the trivial character so the proof is complete. □

Next we deal with unitary groups. Note that $\mathrm{PSU}(2, r) \cong \mathrm{PSL}(2, r)$ which we considered already in Lemma 1.2. With the group $\mathrm{PSU}(3, r)$ was dealt earlier in Lemma 3.7.

Lemma 3.23 $G = \mathrm{PSU}(d, r)$ $(d \geq 4)$ *is not a* TI-*group for characters.*

Proof: According to [18] the action of G on the set of isotropic lines has rank 3. By ([13], 9.17), the number of these lines is

$$n = \frac{r^{2d-1} + (-1)^d(r^d - r^{d-1}) - 1}{r - 1}.$$

Thus, for d even,

$$n = (r^{d-1} + 1)\frac{r^d - 1}{r - 1}$$

and, for d odd,

$$n = (r^d + 1)\frac{r^{d-1} - 1}{r - 1}.$$

Observe that in the first case d and in the second $d - 1$ is not a prime. Thus Lemma 3.14 and Lemma 3.16 imply that n contains at least three different primes except the case $r = 2$ and $d = 4$. But for $\mathrm{PSU}(4, 2)$ the statement follows by the GAP library (see Table 1). An application of Proposition 2.8 finishes the proof. □

Lemma 3.24 *The groups $E_6(r)$, $E_7(r)$, $E_8(r)$, $F_4(r)$, ${}^2E_6(r)$ and ${}^3D_4(r)$ are not* TI*-groups for characters.*

Proof: The listed groups have the following permutation representations of degree n and rank k.

group	degree n	rank k	reference
a) $E_6(r)$	$\frac{r^9-1}{r-1}(r^8+r^4+1)$	3	[26]
b) $E_7(r)$	$\frac{r^{14}-1}{r-1}(r^9+1)(r^5+1)$	4	[26]
c) $E_8(r)$	$\frac{r^{30}-1}{r-1}(r^{12}+1)(r^{10}+1)(r^6+1)$	5	[26]
d) $F_4(r)$	$\frac{r^{12}-1}{r-1}(r^4+1)$	5	[25]
e) ${}^2E_6(r)$	$\frac{r^{12}-1}{r^2-1}(r^5+1)(r^9+1)$	5	[19]
f) ${}^3D_4(r)$	$(r+1)(r^8+r^4+1)$	4	[27]

Let $f(r)$ denote one of the expressions in the second column. We replace r by the indeterminate x and factor the corresponding polynomials $f(x)$ over the rationals in irreducible polynomials. Note that all the obtained factors are cyclotomic polynomials $\mu_d(x)$. By Zsigmondy's Lemma 3.3, a primitive prime divisor of r^d-1 divides $\mu_d(r)$, but not $\mu_t(r)$ for $1 \le t < d$ except of the cases $r=2$ and $d=6$ or $d=2$ and r a Mersenne prime. Moreover, in the second case $\mu_2(r) = r+1 = 2^s$, $\mu_1(r) = r-1 = 2^s-2$ and we still may choose 2 different primes dividing $\mu_2(r)$ and $\mu_1(r)$.

This implies that the number of different irreducible factors not equal to $\mu_6(x)$ is a lower bound for the number of different primes occuring in $f(r)$. In case $r \neq 2$ we may also count the factor $\mu_6(x)$.
The factorizations are as follows

a) $\frac{x^9-1}{x-1}(x^8+x^4+1) = \mu_{12}(x)\mu_9(x)\mu_6(x)\mu_3(x)^2$.

b) $\frac{x^{14}-1}{x-1}(x^9+1)(x^5+1) = \mu_{18}(x)\mu_{14}(x)\mu_{10}(x)\mu_7(x)\mu_6(x)\mu_2(x)^3$.

c) $\frac{x^{30}-1}{x-1}(x^{12}+1)(x^{10}+1)(x^6+1) =$
$$\mu_{30}(x)\mu_{24}(x)\mu_{20}(x)\,\mu_{15}(x)\mu_{12}(x)\mu_{10}(x)\mu_8(x)\mu_6(x)\mu_5(x)\mu_4(x)^2\mu_3(x)\mu_2(x).$$

d) $\frac{x^{12}-1}{x-1}(x^4+1) = \mu_{12}(x)\mu_8(x)\mu_6(x)\mu_4(x)\mu_3(x)\mu_2(x)$.

e) $\frac{x^{12}-1}{x^2-1}(x^5+1)(x^9+1) = \mu_{18}(x)\mu_{12}(x)\mu_{10}(x)\mu_6(x)^2\mu_4(x)\mu_3(x)\mu_2(x)^2$.

f) $(x+1)(x^8+x^4+1) = \mu_{12}(x)\mu_6(x)\mu_3(x)\mu_2(x)$.

Thus in each case we see that the degree n contains at least k different primes except of the case f) with $r=2$. But $G = {}^3D_4(2)$ satisfies $(\Phi_2, \Phi_3) > 1$, by the GAP library (see Table 1), and the assertion follows again by Proposition 2.8. □

Lemma 3.25 *Let $r = 2^{2n+1} > 2$. Then ${}^2F_4(r)$ and the Tits simple group ${}^2F_4(2)'$ are not* TI*-groups for characters.*

Proof: By [27], the group ${}^2F_4(r)$ with $r \neq 2$ has a rank 5 permutation representation of degree

$$n = (r+1)(r^6+1)(r^3+1) = (r+1)\frac{r^{12}-1}{r^3-1} = \mu_{12}(r)\mu_6(r)\mu_4(r)(r+1)^2.$$

If r is exceptional then $r = 8$ and n contains 6 different primes. Thus we may assume that r is not exceptional. Therefore $r + 1$ must contain two different primes. As $r \neq 2$ we get three more different primes by the same argument as in Lemma 3.24 and we are done. The group ${}^2F_4(2)'$ satisfies $(\Phi_2, \Phi_3) > 1$, by the GAP library (see Table 1). $\square$

Lemma 3.26 *The groups $G_2(r)$ for $r > 2$ and $G_2(2)'$ are not* TI*-groups for characters.*

Proof: The group $G_2(2)' \cong \mathrm{PSU}(3, 9)$ was already considered in Lemma 3.23. Thus let $r > 2$.

Suppose that $(\Phi_p, \Phi_q) = 1$ for all primes $p \neq q$. By Proposition 3.5, we may assume that r is exceptional.

For $r = 8$ we have $p = 19 \mid r^2 - r + 1 = 57$ and by [23], we obtain $(\Phi_{19}, \chi_{16}) = 1$ for $\chi_{16} \in \mathrm{Irr}(G_2(8))$. On the other hand, for $q = 3 \mid r + 1 = 9$, we obtain, by ([10], p. 98), that $(\Phi_3, \chi_{16}) = 2$. Thus $(\Phi_3, \Phi_{19}) > 1$, a contradiction to the assumption.

By [25], there is a transitive permutation representation of $G_2(r)$ of degree $n = \frac{r^6-1}{r-1}$ and rank 4. We claim that n contains at least 4 different primes.

Assume first that $r = 2^{2^s}$ with $s > 0$ and $r + 1$ a Fermat prime, so in this case

$$n = \frac{2^{2^s 6} - 1}{2^{2^s} - 1}.$$

If $r = 4$, then

$$n = 3 \cdot 5 \cdot 7 \cdot 13$$

and the claim follows. Now let $s > 1$. By Lemma 3.3, we may take a primitive prime divisor p_1 of $2^{2^s 6} - 1$, a primitive prime divisor p_2 of $2^{2^s 2} - 1$, a primitive prime divisor p_3 of $2^{2^s 3} - 1$ and a prime $p_4 \mid 2^{2^{s-1} 3} - 1$ with $p_4 \nmid 2^{2^s} - 1$. Then all these primes are distinct and $p_1, p_2, p_3, p_4 \mid n$.

Next we consider the case that r is a Mersenne prime, hence $r = 2^l - 1$ with l a prime. By the GAP library (see Table 1), $r \neq 3$, hence l is odd. Note that this implies that $r \equiv 1 \mod 3$ and therefore

$$2, 3 \mid (r + 1)(r^2 - r + 1)(r^2 + r + 1) = n.$$

Let p_1 be a primitive prime divisor of $r^6 - 1$ (note that $r \neq 2$) and let p_2 be a primitive prime divisor of $r^3 - 1$. Then $p_1, p_2 \mid n$ and as $2, 3 \mid r^2 - 1$ we get $p_1, p_2 \neq 2, 3$. Thus we have again 4 different prime divisors of n.

In both cases, i.e. $r + 1$ a Fermat prime or r a Mersenne prime, an application of Proposition 2.8 finishes the proof. $\square$

4 Symmetric and alternating groups

In this section we prove

Proposition 4.1 *Symmetric or alternating groups on $n \geq 5$ letters are not* TI*-groups for characters.*

Note that we only have to prove Propostion 4.1 for symmetric groups because of Lemma 3.21.

For the rest of this section let $G = \mathrm{S}_n$ with $n \geq 5$. By Corollary 2.9 we only have to consider the case when n is a power of a prime and by the GAP library (see Table 2), we may even assume $n > 17$. To deal with the remaining cases we will use a result of James in his book [16] which uses the following notation.

Definition 4.2 [16] Let p be a prime. We say that an integer n contains $\lambda \bmod p$ if in the p-adic representations

$$n = a_s p^s + a_{s-1} p^{s-1} + \ldots + a_1 p + a_0$$
$$\lambda = b_t p^t + b_{t-1} p^{t-1} + \ldots + b_1 p + b_0$$

with $a_s, b_t \neq 0$ we have $t < s$ and for each i either $b_i = 0$ or $b_i = a_i$. We put

$$f_p(n, \lambda) = \begin{cases} 1, & \text{if } n+1 \text{ contains } \lambda \bmod p \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.3 ([16], Theorem 24.15) *Let p be a prime and let $(n-\lambda, \lambda)$ be a 2-part partition of n. Then*

$$[S^{(n-\lambda,\lambda)}, D^{(n)}]_p = f_p(n, \lambda),$$

which means that modulo p the trivial character occurs in the classical irreducible character $\chi^{(n-\lambda,\lambda)}$ associated to the partition $(n-\lambda, \lambda)$ with multiplicity $f_p(n, \lambda)$. In particular, $(\Phi_p, \chi^{(n-\lambda,\lambda)}) \geq f_p(n, \lambda)$.

Lemma 4.4 *Let p be a prime.*

a) If $p > \lambda$ then $f_p(n, \lambda) = 1$ if and only if $p \mid n - \lambda + 1$.

b) If $p^i > \lambda$ for some $i \in \mathrm{N}$ and $p^i \mid n - \lambda + 1$ then $f_p(n, \lambda) = 1$.

Proof: a) Let $p > \lambda$. Thus $n + 1$ contains $\lambda \bmod p$ if and only if $n + 1 \equiv \lambda \bmod p$.
b) If $p^i \mid n - \lambda + 1$ with $p^i > \lambda$ then $n + 1 = kp^i + \lambda$ contains $\lambda \bmod p$. □

As an immediate consequence we have

Lemma 4.5 *If there are primes $p \neq q$ with $p, q > \lambda$ and $p, q \mid n - \lambda + 1$ then*

$$f_q(n, \lambda) = 1 = f_p(n, \lambda)$$

and therefore $(\Phi_p, \Phi_q) > 1$.

Observe that Lemma 4.5 covers again the case $n \neq p^l$ for p a prime with $\lambda = 1$.

Lemma 4.6 *Let $n = 2^l$. Then there are primes $p \neq q$ such that $(\Phi_p, \Phi_q) > 1$.*

Proof: If $l = 2s$ then $n - 1 = (2^s - 1)(2^s + 1)$. Since $s \geq 2$ and $gcd(2^s - 1, 2^s + 1) = 1$ there are odd primes $p \neq q$ with $pq \mid n - 1$. Thus the assertion follows by Lemma 4.5 with $\lambda = 2$. In case $l = 2s + 1$ we have $n - 2 = 2(2^s - 1)(2^s + 1)$. If none of $2^s - 1$ and $2^s + 1$ is power of 3 we are done again by Lemma 4.5 with $\lambda = 3$. Thus we may first suppose that $2^s - 1 = 3^t$. By Lemma 3.2, the only solution is $s = 2$ and therefore $n = 32$. But $n + 1 = 33 = 5^2 + 5 + 3 = 4 \cdot 7 + 5$ contains $\lambda = 5$ modulo the primes 5 and 7, which proves that $f_5(32, 5) = 1 = f_7(32, 5)$. Thus the assertion follows by Theorem 4.3. Finally, we are left with the case $2^s + 1 = 3^t$. Since $s \geq 2$ the only solution is $s = 3$ and $t = 2$, by Lemma 3.2. Thus $n = 2^7$ and $2^7 - 2 = 2 \cdot 7 \cdot 9$. By Lemma 4.4 a), we get $f_7(2^7, 3) = 1$, and by part b) of the same Lemma $f_3(2^7, 3) = 1$. □

Lemma 4.7 *If $n = r^l \equiv 1 \bmod 4$ with r prime then there are primes $p \neq q$ such that $(\Phi_p, \Phi_q) > 1$.*

Proof: Since $4 \mid n - 1$ we have $f_2(n, 2) = 1$, by Lemma 4.4 b). If $n - 1$ is not a power of 2 then there is an odd prime $p \mid n - 1$, hence $f_p(n, 2) = 1$, by part a) of Lemma 4.4. Thus we may assume that $r^l - 1 = 2^s$ with $s \geq 4$. By Lemma 3.2, the solutions of this equality are $r = 3, l = 2, s = 3$ a contradiction since $n > 17$, or $l = 1$ and $r = 2^s + 1$ is a Fermat prime. Thus s is a power of 2, in particular $s = 2t$ with $t \geq 3$. We consider now

$$n - 2 = r - 2 = (2^t - 1)(2^t + 1).$$

If both $2^t - 1$ and $2^t + 1$ are not powers of 3 we are done by Lemma 4.4. Since $t \geq 3$ the equation $2^t - 1 = 3^a$ has no solution. The other equation $2^t + 1 = 3^a$ has, by Lemma 3.2, only the solution $t = 3$ and $a = 2$. But then $n - 2 = r - 2 = 7 \cdot 9$, hence $r = 65$, a contradiction. □

According to Lemmata 4.5, 4.6 and 4.7 we are left with the case $n = r^l \equiv 3 \bmod 4$. In particular, $r \equiv 3 \bmod 4$. If $l = 2s$ then $r^l \equiv (3^2)^s \equiv 1 \bmod 4$ and Lemma 4.7 applies. Thus we may assume that l is odd.

Lemma 4.8 *Let $n = r^l \equiv 3 \bmod 4$ with r prime and l odd and suppose that one of the following conditions holds.*

(i) $n \equiv 0 \bmod 3$

(ii) $n \equiv 1 \bmod 3$

(iii) $n \equiv 2 \bmod 9$.

Then there are primes $p \neq q$ such that $(\Phi_p, \Phi_q) > 1$ unless $n = 19$ or $n = 27$.

Proof: (i) Clearly, $r = 3$ and we may assume $l = 2s+1$ with $s > 1$ since we excluded $n = 27$. If we put $\lambda = 4$ then

$$n - \lambda + 1 = n - 3 = 3^l - 3 = 3(3^s - 1)(3^s + 1).$$

Note that either $4 \mid 3^s + 1$ or $4 \mid 3^s - 1$. Therefore $8 \mid n - 3$ so we have $f_2(n, 4) = 1$ by Lemma 4.4 b). If both $3^s - 1$ and $3^s + 1$ are powers of 2 then $s = 1$ which is a contradiction. Therefore there is a prime $p \geq 5 > \lambda = 4$ such that $p \mid n - 3$. Thus $f_p(n, 4) = 1$, again by

Lemma 4.4, and the proof for (i) is complete.
(ii) Since we assume that $n \equiv 3 \mod 4$ the 2-part of $n-1$ is exactly 2. Thus we are done by Lemma 4.5 with $\lambda = 2$ if $\frac{n-1}{2}$ is not a power of a prime. In case it is a prime power we get

$$n - 1 = 2 \cdot 3^d$$

since $n \equiv 1 \mod 3$. If $d = 2s$ we put $\lambda = 4$ and obtain

$$n - 3 = n - \lambda + 1 = 2(3^s - 1)(3^s + 1).$$

Since $8 \mid n-3$ we have $f_2(n, 4) = 1$. And as before $s > 1$ because if $s = 1$ we get $n = 19$. Therefore there is a prime $p \geq 5 > \lambda = 4$ with $f_p(n, 4) = 1$. Finally, if $d = 2s + 1$ we put $\lambda = 8$. Then

$$n - 7 = n - \lambda + 1 = 6(3^s - 1)(3^s + 1).$$

If $s = 1$ then $n = 55$ and we are done by Lemma 4.5 with $\lambda = 1$, thus $s \geq 2$. Note that $16 \mid n - 7$ and therefore $f_2(n, 8) = 1$, by Lemma 4.4 b). By the same lemma we are also done if 5^2 or 7^2 or a prime $q > 7$ divides $n - 7$. This shows that we may assume

$$(3^s - 1)(3^s + 1) = 5^a 7^b 2^c$$

with $a, b \leq 1$. Since the 2-part of one of the factors $3^s - 1$ or $3^s + 1$ is exactly 2 this factor is 2, 10, 14 or 70, but only 10 is of the form $3^s \pm 1$ with $s > 1$. Hence $s = 2$ which implies $n = 487$ which is a prime. For $\lambda = 3$, we have $487 - \lambda + 1 = 487 - 2 = 5 \cdot 97$ and the assertion follows by Lemma 4.4.
(iii) If $9 \mid n-2$ then $9 \mid n - \lambda + 1$ for $\lambda = 3$. Thus $f_3(n, 3) = 1$, by Lemma 4.4 b). Since $n-2$ is odd, we may apply again Lemma 4.4 if there exists a prime $q \geq 5$ with $q \mid n - 2$ which yields $f_q(n, 3) = 1$. Therefore we only have to consider the case $n - 2 = 3^t$ with $t \geq 3$. If $t = 2s$ we choose $\lambda = 4$. Then

$$n - \lambda + 1 = n - 3 = 3^{2s} - 1 = (3^s - 1)(3^s + 1).$$

Since $8 \mid n - 3$, we get $f_2(n, 4) = 1$, by Lemma 4.4 b). Furthermore $(3^s - 1)(3^s + 1)$ is not a power of 2 since $s > 1$. Thus there is a prime $q > \lambda = 4$ with $q \mid n - 3$ and therefore $f_q(n, 4) = 1$, by Lemma 4.4. Finally we have to consider $t = 2s + 1$. With $\lambda = 6$ we get

$$n - \lambda + 1 = n - 5 = 3^{2s+1} - 3 = 3(3^s - 1)(3^s + 1).$$

If $s = 1$ then $n = 29$. Since $n - 1 = 28 = 4 \cdot 7$ we get by Lemma 4.4 that $f_2(29, 2) = 1 = f_7(29, 2)$. So we may assume $s \geq 2$. Clearly, $8 \mid n-5$, hence $f_2(n, 6) = 1$. Then we are done by Lemma 4.4 if there is a prime $q > 5$ with $q \mid n - 5$. Otherwise, either $5^2 \mid n - 5$ (so n is a power of 5) or one of $3^s \pm 1$ has to be a 2-power. But $3^s + 1$ is never a 2-power for $s \geq 2$, and if $3^s - 1$ is a 2-power then $s = 2$ and $n = 3^5 + 2 = 245$, which is not a prime power, a contradiction. Finally, if $5^2 \mid n - 5$ then Lemma 4.4 b) yields $f_5(n, 6) = 1$. □

To finish the proof of Proposition 4.1 we consider 3-part partitions. The relevant fact which we need is the following result of To Law.

Lemma 4.9 (Appendix, [17]) *If* $\Gamma = (n - \lambda - 1, \lambda, 1)$ *then*

$$[S^\Gamma, D^{(n)}]_p \geq \left(\sum_{j=0}^{\lambda} m_j f_p(n - 1 - 2j, \lambda - j) \right) - f_p(n, \lambda) - f_p(n, \lambda + 1)$$

for suitable $m_j \in \mathrm{N}_0$ which are given explicitely in [17]*. In particular (we quote only the cases that will be used below):*

α) If $n \not\equiv 0$ mod p, then $m_0 = 1$,

β) If $1 \leq j \leq p-2$ and $n-j \equiv -1$ mod p, then $m_j = 1$,

γ) If there are integers c, u, v, k with $c, v > 0$, $1 \leq u \leq p-1$ and $-1 \leq k \leq p^v - 2$ such that $j = u \cdot p^v - 1$ and $n - j = c \cdot p^{v+1} + k$, then $m_j = 2$.

In the next two lemmas we prove the proposition for $n = 19$ and $n = 27$ which we had to exclude in Lemma 4.8.

Lemma 4.10 *For $n = 19$ we have $(\Phi_5, \Phi_2) > 1$.*

Proof: Let $\Gamma = (14, 4, 1)$. First we consider $p = 5$. By Lemma 4.9, we have

$$[S^\Gamma, D^{(19)}]_5 \geq m_0 f_5(18, 4) - f_5(19, 4) - f_5(19, 5).$$

Now, by Lemma 4.4, the equation $19 = 18 + 1 = 3 \cdot 5 + 4$ implies $f_5(18, 4) = 1$, and $20 = 19 + 1 = 4 \cdot 5$ implies $f_5(19, 4) = 0 = f_5(19, 5)$. Furthermore, $m_0 = 1$, by 4.9 α) which proves that $[S^\Gamma, D^{(19)}]_5 \geq 1$.

For $p = 2$ we have

$$[S^\Gamma, D^{(n)}]_2 \geq m_3 f_2(12, 1) - f_2(19, 4) - f_2(19, 5).$$

The equation $13 = 12 + 1 = 2^3 + 2^2 + 1$ yields $f_2(12, 1) = 1$. Since $20 = 19 + 1 = 2^4 + 2^2$ we obtain $f_2(19, 4) = 1$ and $f_2(19, 5) = 0$. By Lemma 4.9 γ) , we also have $m_3 = 2$ (since $j = 3 = 2^v - 1$ and $n - j = 16 = 2 \cdot 2^{v+1} + k$ with $c = 2$, $u = 1$, $v = 2$ and $k = 0$). This shows that $[S^\Gamma, D^{(19)}]_2 \geq 1$ and the proof is complete. □

Lemma 4.11 *For $n = 27$ we have $(\Phi_3, \Phi_2) > 1$.*

Proof: We put $\Gamma = (19, 7, 1)$. For $p = 3$ we have

$$[S^\Gamma, D^{(n)}]_3 \geq m_1 f_3(24, 6) - f_3(27, 7) - f_3(27, 8).$$

The equation $28 = 27 + 1 = 3^3 + 1$ yields $f_3(27, 7) = f_3(27, 8) = 0$. Since $25 = 24 + 1 = 2 \cdot 3^2 + 2 \cdot 3 + 1$ we get $f_3(24, 6) = 1$. We have $j = 1 \leq p - 2 = 1$ and $n - j = 26 \equiv -1$ mod 3, hence, by Lemma 4.9 β), $m_1 = 1$. Thus $[S^\Gamma, D^{(n)}]_3 \geq 1$.

In case $p = 2$ we have

$$[S^\Gamma, D^{(n)}]_2 \geq m_3 f_2(20, 4) - f_2(27, 7) - f_2(27, 8).$$

The equation $28 = 27 + 1 = 2^4 + 2^3 + 2^2$ yields $f_2(27, 7) = 0$ and $f_2(27, 8) = 1$. Furthermore, $f_2(20, 4) = 1$ since $21 = 20 + 1 = 2^4 + 2^2 + 1$. By Lemma 4.9 γ), we obtain $m_3 = 2$ as $3 = j = u2^v - 1$ and $n - j = 24 = 3 \cdot 2^{v+1} + k$ with with $c = 2$, $u = 1$, $v = 2$ and $k = 0$. This shows that $[S^\Gamma, D^{(n)}]_2 \geq 1$. □

Now we are ready to finish the proof of Proposition 4.1.

Lemma 4.12 *Let* $n \equiv 2 \bmod 3$, $n \not\equiv 2 \bmod 9$ *and* $n \equiv 3 \bmod 4$. *Then* $(\Phi_2, \Phi_3) > 1$.

Proof: Let $\Gamma = (n-3, 2, 1)$.

Consider first the case $p = 3$. As $n+1 \equiv 0 \bmod 3$ and $n+1 \not\equiv 3 \bmod 9$ we have either $n+1 = 9k+6$ or $n+1 = 9k$. In both cases $n+1$ does not contain neither 2 nor 3 mod 3. Therefore $f_3(n,2) = f_3(n,3) = 0$. But $f_3(n-1,2) = 1$ as $3 \mid n-2$. Thus we obtain, using Lemma 4.9 α) for the last equality

$$[S^\Gamma, D^{(n)}]_3 \geq m_0 f_3(n-1,2) - f_3(n,2) - f_3(n,3) = m_0 = 1.$$

Now, for $p = 2$, since $n+1 \equiv 0 \bmod 4$ we have that $n+1$ does not contain neither 2 nor 3 mod 2, hence $f_2(n,2) = f_2(n,3) = 0$. But $n = 4k-1 = 4(k-1)+2+1$ contains 2 mod 2, hence $f_2(n-1,2) = 1$. Thus we get again by Lemma 4.9 α)

$$[S^\Gamma, D^{(n)}]_2 \geq m_0 f_2(n-1,2) - f_2(n,2) - f_2(n,3) = m_0 = 1.$$

□

5 Sporadic groups

Proposition 5.1 *A sporadic simple group is not a* TI*-group for characters.*

Proof: If G is not one of the groups $J_4, Th, Fi'_{24}, B, M, Ly$ or Co_1 we find decomposition matrices in the GAP library to check the assertion. If G is one of the groups J_4, Th, M decomposition matrices for cyclic blocks are given in [10] and the assertion follows by using Proposition 2.6. Finally, the same approach works for Fi'_{24} and B. In both cases the decomposition matrices for cyclic blocks of the cover groups are given in [10]. But the relevant characters are already characters of the simple groups. For specific primes and characters for all these groups, see Table 1. The assertion for the remaining two groups Co_1 and Ly follows by recent results of J. Thackray which was brought to our attention by Lukas Mass from IEM at Essen. More precisely, the Conway group Co_1 has an irreducible character χ_{10} of degree 80730 which is contained in Φ_2 and Φ_7. The Lyons group has a real valued character χ_{14} of degree 4226695 which contains the trivial character modulo 3. Thus $(\Phi_3, \Phi_2) > 1$, by Proposition 2.6. □

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Table 1: Primes with non trivial intersection for some simple groups. (from the GAP library [7] or [10]).

Group	Primes p, q	Character $\chi \in \Phi_p \cap \Phi_q$
$\mathrm{PSL}(3,3)$	2,3	χ_{12}
$\mathrm{PSL}(5,2)$	2,3	χ_6
$\mathrm{P\Omega}^+(8,2)$	2,3	χ_7
$\mathrm{P\Omega}^+(10,2)$	3,5	χ_6
$\mathrm{P\Omega}^-(8,2)$	2,3	χ_3
$\mathrm{PSU}(4,2)$	2,3	χ_8
$\mathrm{PSL}(3,8)$	2,7	χ_3
$\mathrm{PSp}(6,2)$	2,3	χ_3
${}^3D_4(2)$	2,3	χ_{10}
${}^2F_4(2)'$	2,3	χ_{14}
$G_2(3)$	2,3	χ_7
HN	2,5	χ_9
HS	2,3	χ_{20}
He	2,3	χ_{26}
J_1	2,3	χ_6
J_2	2,3	χ_{20}
J_3	2,3	χ_2
J_4	2,7	χ_8
Ly	2,3	χ_{14}
M_{11}	2,3	χ_5
M_{12}	2,3	χ_2
M_{22}	2,3	χ_7
M_{23}	2,3	χ_{17}
M_{24}	2,3	χ_2
McL	2,3	χ_9
ON	2,3	χ_8
Ru	2,3	χ_{23}
Suz	2,3	χ_2
Co_1	2,7	χ_{10}
Co_2	2,3	χ_{14}
Co_3	2,3	χ_2
Fi_{22}	2,3	χ_3
Fi_{23}	2,5	χ_{26}
Fi'_{24}	2,13	χ_8
Th	2,31	χ_7
B	2,17	χ_{28}
M	2,31	χ_{11}

Table 2: Primes with non trivial intersection for some symmetric groups S_n where n is a power of a prime (from the GAP library [7]).

Group	Primes p, q	Partition π with $\chi_\pi \in \Phi_p \cap \Phi_q$
S_5	2,3	$1^1 2^2$
S_7	2,3	$1^1 2^1 4^1$
S_8	2,3	$1^2 3^2$
S_9	2,3	$1^1 2^1 6^1$
S_{11}	2,3	$1^1 2^1 8^1$
S_{13}	2,3	$2^1 11^1$
S_{16}	2,3	$1^2 2^1 12^1$
S_{17}	2,3	$1^1 2^1 14^1$