On Degrees of Irreducible Brauer Characters

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1 Introduction

Let G be a finite group and let $\mathrm{IBr}_p(G)$, resp. $\mathrm{IBr}_p(B)$ be the set of irreducible p-Brauer characters of G, resp. of a p-block B. For $\varphi \in \mathrm{IBr}_p(G)$ let Φ_φ denote the projective indecomposable character corresponding to φ. Due to Brauer and Nesbitt [6] the term $|G| - \frac{|G|}{\Phi_1(1)}$ where Φ_1 denotes the projective character corresponding to the trivial character is an upper bound for the dimension of the Jacobson radical of the p-modular group algebra of G. An obvious reformulation of this result leads to

$$\frac{|G|}{\Phi_1(1)} \leq \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2. \tag{1}$$

Moreover, equality holds if and only if G has a normal Sylow p-subgroup. This was proved by Wallace in [18] for p-solvable groups, and in full generality by Brockhaus in [7] using the classification of finite simple groups.

In case $p = 2$ and $\Phi_1(1) = |G|_p$ where $|G|_p$ is the largest p-part dividing $|G|$ the classification can be avoided to show the normality of a Sylow p-subgroup if equality holds in (1). The condition $\Phi_1(1) = |G|_p$ implies $\Phi_\varphi = \varphi\Phi_1$ since Φ_φ is a constituent of $\varphi\Phi_1$ and $|G| = \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)\Phi_\varphi(1)$. Now, if $p \mid \varphi(1)$, the trivial character has multiplicity at least 2 in $\varphi\overline{\varphi}$ and a contradiction follows considering the scalar product relation

$$1 = (\varphi, \Phi_\varphi) = (\varphi, \varphi\Phi_1) = (\varphi\overline{\varphi}, \phi_1) \geq 2$$

(see [13], VII, 8.5 d)). Thus p does not divide any irreducible Brauer character degree and a nice argument of Okuyama (siehe [16], Theorem 2.33) implies that for $p = 2$ the Sylow p-subgroup of G is normal.

The lower bound in (1) has been improved in [14] replacing $\Phi_1(1)$ by the spectral radius $\rho(C)$ of the Cartan matrix C, i.e. the maximum value $|\lambda|$ where λ runs through the set of complex eigenvalues of C.

However, for obvious reasons we are interested in a lower bound which only depends on terms of G. This is a weaker question than Brauer asked in Problem 15 of his famous list [5]. He actually wanted to have a characterization of the dimension of the Jacobson radical by group theoretical properties.

In case of p-solvable groups (the existence of a p-complement suffices) we have $\Phi_1(1) = |G|_p$ (see [11], Chap X, 3.2). Hence the desired bound is

$$|G|_{p'} \leq \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2. \tag{2}$$

But for non p-solvable groups, $\Phi_1(1)$ resp. $\rho(C)$ may differ extremely from $|G|_p$. For instance, let $G = \mathrm{R}(q)$ with $q = 3^{2m+1}$ $(m \geq 1)$ be a Ree group and let $p = 2$. Then $\Phi_1(1) = 2(q^3+1) \gg 8$ and $\rho(C) \approx 16.38$ (see [15] for the Cartan matrix), but $|G|_2 = 8$ and furthermore

$$\frac{|G|}{8} = |G|_{2'} \leq \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2.$$

(In order to prove the inequality it is enough to consider the $\frac{q-3m}{6}$ characters of defect zero of degree $(q^2-1)(q+1+3m)$, see [19].) Nevertheless for sporadic groups we have checked so far the bound in (2) holds true. The same bound turns out to be true if the Sylow p-subgroups of G are cyclic [14]. So for any finite group G we are led to the

Conjecture. *We always have*

$$|G|_{p'} \leq \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2, \tag{3}$$

and equality holds if and only if G has a normal Sylow p-subgroup.

Note that the *if part* is trivial since a normal p-subgroup is always contained in the kernel of an irreducible representaion in characteristic p.

Suppose that the Conjecture has an affirmative answer. Thus in the extreme case $|G|_{p'} = \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2$ the group G has a normal Sylow p-subgroup, say P. By Lemma 4.26 of ([11], Chapter IV), we know that

$$\Phi_\varphi(x) = |C_P(x)|\varphi(x)$$

for p'-elements $x \in G$. On the other hand, by Lemma 3.8 of ([11], Chapter IV), we have

$$\sum_{\varphi \in \mathrm{IBr}_p(G)} \Phi_\varphi(x)\overline{\varphi(y)} = \begin{cases} 0 & \text{if}\, x \not\sim y \\ |C_G(x)| & \text{if}\, x \sim y \end{cases}$$

for p'-elements $x, y \in G$ where $\sim$ denotes conjugation in G. Thus we get the relation

$$\sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(x)\overline{\varphi(y)} = \begin{cases} 0 & \text{if } x \not\sim y \\ \frac{|C_G(x)|}{|C_P(x)|} & \text{if } x \sim y. \end{cases}$$

So we may ask the

Question. *Is it even true that for p'-elements $x \in G$ we always have*

$$\frac{|C_G(x)|}{|G|_p} \leq \sum_{\varphi \in \mathrm{IBr}_p(G)} |\varphi(x)|^2 \ ? \tag{4}$$

If so, the bound may be sharp for $x \neq 1$ even if the Sylow p-subgroup is not normal. For instance, if $p = 5$ and x is an element of order 3 in $G = \mathrm{L}_2(16)$ then

$$3 = \frac{15}{5} = \frac{|C_G(x)|}{|G|_5} = \sum_{\varphi \in \mathrm{IBr}_5(G)} \varphi(x)\overline{\varphi(x)} = 1^2 + 1^2 + (-1)^2 = 3.$$

Note that the inequality (4) reduces to that in (3) for $x = 1$.

We are furthermore attempted to ask a p-local version of (3). In this case we may replace

$$|G|_{p'} = \frac{|G|}{p^a} = \frac{\dim KG}{p^a}$$

by $\frac{\dim B}{p^d}$ where B is a p-block of defect d. Thus we pose the question whether

$$\frac{\dim B}{p^d} \leq \sum_{\varphi \in \mathrm{IBr}_p(B)} \varphi(1)^2 \tag{5}$$

holds true.

An affirmative answer to (5) was given by Kiyota and Wada in [14] in case G is p-solvable or B is a p-block with cyclic defect group. In the latter case equality holds if and only if the Brauer tree is a star and all irreducible Brauer characters have the same degree.

Unfortunately the principal 2-block B_0 of the alternating group A_5 shows that inequality (5) fails to be true in general because

$$11 = \frac{\dim B_0}{2^d} > \sum_{\varphi \in \mathrm{IBr}_p(B)} \varphi(1)^2 = 1 + 2^2 + 2^2 = 9.$$

A more advanced counterexample is the non-principal 3-block of maximal defect of $2.\mathrm{A}_7$. This example was brought to my attention by Thomas Breuer.

2 On the Conjecture

In this section we prove the Conjecture for groups which have a projective character with properties similiar to those of the Steinberg character for groups of Lie type. We start with a result due to Alperin. Let Φ be any projective character of a finite group G and let $\mathrm{IBr}(G) = \{\varphi_1, \varphi_2, \ldots, \varphi_s\}$. Then Φ uniquely determines a matrix $A = (a_{ij})$ by the equations

$$\varphi_i \Phi = \sum_{j=1}^{s} a_{ij} \Phi_j \qquad (i = 1, \ldots, s).$$

If $x_1, ..., x_s$ are representatives of the p'-conjugacy classes of G and if C denotes the Cartan matrix of G then

Lemma 2.1 [1]

$$\det A = \frac{\prod_{i=1}^{s} \Phi(x_i)}{\det C}.$$

Proof. For $C = (c_{ij})$, we have

$$\varphi_i \Phi = \sum_{j,k=1}^{s} a_{ij} c_{jk} \varphi_k.$$

Therefore we get the matrix equation

$$(\varphi_i(x_l)) \begin{pmatrix} \Phi(x_1) & 0 & \cdots & 0 \\ 0 & \Phi(x_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \Phi(x_s) \end{pmatrix} = AC(\varphi_i(x_l)).$$

Since $(\varphi_i(x_l))$ and C are invertible ([11], Chap. IV, 3.6), the assertion follows.

Theorem 2.2 *Let G be a finite group. Suppose that G has a projective character Φ satisfying the following two conditions.*

(a) $\Phi(1) = |G|_p$.

(b) $\Phi(x) \neq 0$ *for all p'-elements $x \in G$* .

Then

$$|G|_{p'} \leq \sum_{i=1}^{s} \varphi_i(1)^2,$$

and equality holds if and only if G has normal Sylow p-subgroup.

Proof. Condition (b) implies that $\det A \neq 0$ in the above lemma. Thus there exists a permutation π of $\{1, ..., s\}$ such that

$$\prod_{i=1}^{s} a_{\pi i,i} \neq 0.$$

This means $\varphi_{\pi i}\Phi = \Phi_i + \cdots$
If ρ denotes the regular character of G, then

$$\sum_{i=1}^{s} \varphi_i(1)\varphi_{\pi i}\Phi = \sum_{i=1}^{s} \varphi_i(1)\Phi_i + \cdots = \rho + \cdots$$

In particular we have

$$\sum_{i=1}^{s} \varphi_i(1)\varphi_{\pi i}(1)\Phi(1) \geq \rho(1) = |G| \tag{6}$$

and by condition (a)

$$\sum_{i=1}^{s} \varphi_i(1)\varphi_{\pi i}(1) \geq |G|_{p'}. \tag{7}$$

Applying Cauchy-Schwarz's inequality yields

$$(\sum_{i=1}^{s} \varphi_i(1)\varphi_{\pi i}(1))^2 \leq \sum_{i=1}^{s} \varphi_i(1)^2 \sum_{i=1}^{s} \varphi_{\pi i}(1)^2, \tag{8}$$

hence

$$\sum_{i=1}^{s} \varphi_i(1)\varphi_{\pi i}(1) \leq \sum_{i=1}^{s} \varphi_i(1)^2. \tag{9}$$

Putting (7) and (9) together we obtain the desired assertion

$$|G|_{p'} \leq \sum_{i=1}^{s} \varphi_i(1)^2.$$

So it remains to prove that $|G|_{p'} = \sum_{i=1}^{s} \varphi_i(1)^2$ implies that a Sylow p-subgroup of G is normal.

By (6) we get $\varphi_{\pi i}\Phi = \Phi_i$ for all i. In particular we have $\varphi_{\pi 1}\Phi = \Phi_1$ for the trivial character φ_1. The equality in the Cauchy-Schwarz inequality (see (8)) forces $\varphi_{\pi 1}(1) = \varphi_1(1) = 1$. Thus we obtain $\Phi_1(1) = |G|_p$ and by a result of Brockhaus [7] the group G has a normal Sylow p-subgroup.

The properties we required on the projective character in 2.2 seem to be very strong. However there are interesting examples in which the Theorem applies.

Examples 2.3 a) Let G be a finite group with a split BN-pair of characteristic p and satisfying the commutator relations (see [9]). For Φ we take the Steinberg character. Then condition a) and b) are satisfied since

$$\Phi(x) = |C_G(x)|_p$$

for all p'-elements x of G (see [9], 6.4.7). Note that for Chevalley groups (twisted or non-twisted) defined in characteristic p the square of the degree of the Steinberg character already dominates the p'- part of the order of G.
b) Let G be a finite group with a p-complement. Now we may take for Φ the projective indecomposable character corresponding to the trivial character. Observe that Φ is the trivial character of a p-complement induced to G and satisfies the assumptions of 2.2.
c) More subtle is the following. Let N be a normal p-solvable subgroup of G and suppose that $H = G/N$ possess a projective character satisfying condition a) and b). We prove that G has a projective character with the same properties. So let $M \leq N$ be a minimal normal subgroup of G. By the inductive hypothesis the group G/M has a projective character, say Ψ, satisfying properties a) and b). If M is a p'-group we may take for Φ the inflation $\Phi = \mathrm{infl}_G(\Psi)$ of Ψ. Thus suppose that M is a p-group. Let λ be the Brauer character of the conjugation action of G on M. Now we put

$$\Phi = \lambda \ \mathrm{infl}_G(\Psi).$$

That ϕ is projective follows by a result of Alperin, Collins and Sibley (see [2]). Clearly,

$$\phi(1) = \lambda(1) \ \mathrm{infl}_G(\Psi(1)) = |M||H| = |G|_p$$

and

$$\phi(x) = \lambda(x) \ \mathrm{infl}_G(\Psi(x)) = |C_M(x)| \ \mathrm{infl}_G(\Psi(x)) \neq 0$$

for all p'-elements $x \in G$.

3 Asymptotic results for groups of Lie type

To give more evidence for inequality (3) we will look in this section asymptotically on groups of Lie type in non-defining characteristics. For a detailed introduction to the character theory of groups of Lie type the reader is referred to [9].

Let G be a simple linear algebraic group over an algebraically closed field K of positive characteristic $r \neq p$ and let $G^F = G(q)$ (where $q = r^f$) denote the group of fixed points under a Frobenius map F. Furthermore let $T_w = T^F$ be the maximal torus in G^F corresponding to $w \in W = W(T)$ where W denotes the Weylgroup of an F-stable maximal torus $T \subseteq G$.

Now we look at irreducible Deligne-Lusztig characters $R_{T,\theta}$ (up to a sign) where $\theta \in \widehat{T}^F = \mathrm{Hom}\,(T^F, \mathbb{C}^*)$ is an irreducible character of $T_w = T^F$ and θ

is in general position. Let $\mathcal{R}(q)$ denote the set of all such $R_{T,\theta}$'s. Note that $R_{T,\theta} = R_{T,\theta'}$ if and only if $\theta = {}^{w'}\theta'$ for some

$$w' \in W(T)^F = C_{w,F} = \{w' \in W \mid {w'}^{-1} w F(w') = w\}$$

(see [9], p. 219). Thus the number $|\mathcal{R}(q)|$ of different $R_{T,\theta}$'s is equal to the number of $W(T)^F$-orbits of characters θ in general position. By a result of Veldkamp ([17]) this number is equal to the number of $W(T)^F$-conjugacy classes of elements in general position in the fixed point group ${T^*}^{F^*}$ of the dual torus $T^* \subseteq G^*$.

For simplicity let us suppose that G^* is simply connected. In this case the regular elements in ${T^*}^{F^*}$ coincide with the elements in general position. (Note that $t \in G$ is regular resp. in general position if $C_G(t)^\circ = T$ resp. $C_G(t) = T$.) Let l denote the Lie rank of G. Applying 3.1 and 3.2 in [10] (see also [12]) we get

$$|\mathcal{R}(q)| = \{R_{T,\theta} \mid \theta \text{ in general position}\}| = \frac{q^l}{|C_{w,F}|} + O(q^{l-1}).$$

Note that $R_{T,\theta}$ is of degree

$$R_{T,\theta}(1) = \frac{|G(q)|_{r'}}{|T_w|}.$$

Thus we obtain

$$\sum_{R_{T,\theta} \in \mathcal{R}(q)} R_{T,\theta}(1)^2 = \left(\frac{q^l}{|C_{w,F}|} + O(q^{l-1}) \right) \frac{|G(q)|_{r'}^2}{|T_{[w]}|^2}. \tag{10}$$

Let N denote the number of positive roots. As polynomials in q we have

- $|T_w| = q^l +$ lower terms in q,
- $|G(q)|_{r'} = q^{l+N} +$ lower terms in q,
- $|G(q)| = q^{l+2N} +$ lower terms in q.

Inserting these equations in (10) we obtain

$$\begin{aligned} \textstyle\sum_{R_{T,\theta} \in \mathcal{R}(q)} R_{T,\theta}(1)^2 &= \tfrac{1}{|C_{w,F}|} q^{l+2N} + O(q^{l+2N-1}) & (11) \\ &= |G(q)|_{p'} + (\tfrac{1}{|C_{w,F}|} - \tfrac{1}{|G(q)|_p}) q^{l+2N} + O(q^{l+2N-1}). \end{aligned}$$

We may assume that $p \mid |G(q)|$. If we exclude the Steinberg triality ${}^3D_4(q^2)$ for a moment then

$$|G(q)|_{r'} = \prod_{i=1}^{l} (q^{d_i} - \epsilon_i)$$

where $\epsilon_i \in \{1,-1\}$ and the d_i are the exponents of the underlying group (see [9], p. 75). Thus $p \mid q^{2d_i}-1$. Note that

$$(q^{2d_i}-1)_p < (q^{2pd_i}-1)_p$$

and $|W|$ is independent of q (see [3], Planche II). Thus there exists an $s \in \mathbb{N}$ (for instance s a suitable power of p) such that

$$|C_{w,F}| \leq |W| < |G(q^s)|_p$$

and therefore

$$|C_{w,F}| < |G(q^{st})|_p$$

for all $t \in \mathbb{N}$ and all $w \in W$. This is also true for ${}^3D_4(q^2)$ as a similar argument shows. By (11) we obtain

$$\sum_{R_{T,\theta} \in \mathcal{R}(q^{st})} R_{T,\theta}(1)^2 \geq |G(q^{st})|_{p'}$$

for all $t \geq t_0$. Now, if there exist a torus $T_w \subseteq G(q^{st})$ with $p \nmid |T_w|$ then all characters $R_{T,\theta}$ with θ in general position are irreducible and of p-defect zero. In particular they are irreducible Brauer characters for the prime p and we have the desired inequality

$$\sum_{\varphi \in \mathrm{IBr}_p(G(q^{st}))} \varphi(1)^2 \geq |G(q^{st})|_{p'}$$

for all $t \geq t_0$. Thus we have proved

Theorem 3.1 *Let $G(q) = G^F$ be a finite group of Lie type where G is a linear simple algebraic group and G^* is simply connected. Let p be a prime with $p \nmid q$ and $p \mid |G(q)|$.*

a) Then there exist an $s \in \mathbb{N}$ such that

$$\sum_{R_{T,\theta} \in \mathcal{R}(q^{st})} R_{T,\theta}(1)^2 \geq |G(q^{st})|_{p'}$$

for all $t \geq t_0$.

b) If $p \nmid |T_w|$ for some $w \in W$ then

$$\sum_{\varphi \in \mathrm{IBr}_p(G(q^{st}))} \varphi(1)^2 \geq |G(q^{st})|_{p'}$$

for all $t \geq t_0$.

The critical point in the whole process is the assumption $p \nmid |T_w|$ for a suitable $w \in W$. For instance, if $G(q) = \mathrm{PGL}(2,q)$ we can not find such a torus for $p = 2$. Thus our method fails for $p = 2$.

The following example may illustrate the above.

Example Let $G^F = G(q) = \mathrm{PGL}(l+1,q)$ be the finite adjoint group of type A_l for $l \geq 2$ with q a power of r. Note that $G^{*F^*} = \mathrm{SL}(l+1,q)$ is the fixed point group of a simply connected group. Let $w \in W$ be a Coxeter element, i.e. w is a cycle of length $l+1$ and $C_{w,F} = C_{\mathrm{S}_{l+1}}(w) = \langle w \rangle$ where S_n is the symmetric group on n letters. Let p be a prime with $p > l+1$. Thus we either have $p \nmid |G(q)|$ and (3) holds with equality or $|C_{w,F}| < |G(q)|_p$ and we have $\sum_{R_{T,\theta} \in \mathcal{R}} R_{T,\theta}(1)^2 \geq |G(q)|_{p'}$ for $q \geq q_0$ and $p \mid |G(q)|$. Given such a q. Now if $p \nmid |T_w| = 1 + q + \ldots + q^l$, then

$$\sum_{\varphi \in \mathrm{IBr}_p(G(q))} \varphi(1)^2 \geq |G(q)|_{p'}.$$

The condition $p \nmid |T_w|$ is satisfied for instance if $p \nmid q-1$ and $p \mid q^s - 1$ where $2 < s < l+1$ and $\gcd(s, l+1) = 1$.
Let us specialize to $G(q) = \mathrm{PGL}(2,q)$ and let $2 < p \neq r$, hence $2 = |W| < p$. Since $|G(q)| = q(q+1)(q-1)$ we have $p \nmid |T_w| = (q+1)$ or $p \nmid |T_1| = q-1$ where T_w is the Coxeter and T_1 the split torus. Thus for all q large enough and all odd primes inequality (3) holds true for $\mathrm{PGL}(2,q)$. However in this small case we get (3) for any q. If $p = r$ we may apply 2.3 a). In case $2 \neq p \neq r$ the Sylow p-subgroups of $G(q)$ are cyclic. Thus the inequality (5) holds true for every p-block by [14] which obviously implies (3). For $p = 2$ one easily checks inequality (3) by using the results of section VIII in [8].

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