On q-Steiner systems from rank metric codes

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Abstract

In this paper we prove that rank metric codes with special properties imply the existence of q-analogs of suitable designs. More precisely, we show that the minimum weight vectors of a $[2d, d, d]$ dually almost MRD code $C \leq \mathbb{F}_{q^m}^{2d}$ $(2d \leq m)$ which has no code words of rank weight $d+1$ form a q-Steiner system $S(d-1, d, 2d)_q$. This is the q-analog of a result in classical coding theory and it may be seen as a first step to prove a q-analog of the famous Assmus-Mattson Theorem.

Keywords: Rank metric code, q-analog Steiner system, dually AMRD code

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1 Introduction

The interest in q-analogs of codes and designs has been increased over the last years due to their applications in random network coding. It is well-known that a network code $\mathcal{C}$ consisting of k-dimensional subspaces of a fixed n-dimensional vector space over $\mathbb{F}_q$ with minimum distance $d \geq 2(k-t+1)$ has largest size $|\mathcal{C}|$ if and only if the codewords form an $S(t, k, n)_q$ Steiner system, the q-analog of an $S(t, k, n)$ Steiner system. This is one of the reasons we are interested in the existence and construction of such systems.

Definition 1.1. Let $t, k, n, \lambda \in \mathbb{N}$ with $t \leq k \leq n$. A t-$(n, k, \lambda)_q$ subspace design $\mathcal{D}$ over the field $\mathbb{F}_q$ is a set of k-dimensional subspaces of $\mathbb{F}_q^n$, called the blocks, such that every t-dimensional subspace of $\mathbb{F}_q^n$ is contained in exactly λ blocks of $\mathcal{D}$. In case $\lambda = 1$, $\mathcal{D}$ is called a q-Steiner system and denoted by $S(t, k, n)_q$.

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Clearly $S(k,k,n)_q$ and $S(1,n,n)_q$ exist and these are called trivial q-Steiner systems. Furthermore there are non-trivial q-Steiner systems $S(1,k,n)_q$ whenever $k \mid n$ which are also known as spreads [4]. Apart from these examples only $S(2,3,13)_2$ is known to exist [2]. One of the most challenging problems in this field is the question of the existence of an $S(2,3,7)_q$ design, as it would have the smallest parameters of a non-trivial q-Steiner system with $t \geq 2$. It is known as a q-analog of the Fano plane and many authors have been studying this questions so far. See e.g. [3],[17] and the references in these papers.

In this work we analyze q-analogs of Steiner systems derived from a special family of rank metric codes called dually almost MRD codes which were defined in [7]. In Section 2 we collect some facts on rank metric codes, in particular on generalized rank weights. In Section 3 we analyze the supports of the minimum weight vectors of a rank metric code. Section 4 deals with a relationship between rank metric codes and subspace designs. We prove that the minimum weight vectors of a $[2d,d,d]$ dually almost MRD code $C \leq \mathbb{F}_{q^m}^{2d}$ ($2d \leq m$) which has no code words of rank weight $d+1$ hold an $S(d-1,d,2d)_q$ Steiner system. The blocks are built by the supports of all code words of minimum weight. Note that such a code provides

a) for $d=4$ a q-analog of the Fano plane (see Lemma 4.2),

b) for $n=2d$ and $t=d-1$ an affirmative answer of the following question which may be seen as the q-analog of the Assmus-Mattson theorem.

Question 1.2. Let $C \leq \mathbb{F}_{q^m}^n$ be an $\mathbb{F}_{q^m}$-linear code with minimum distance d. Fix a positive integer t with $t < d$ and let s be the number of i with $A_i(C^\perp) \neq 0$ for $0 < i \leq n-t$. Is it then true, that the supports of vectors of weight d in C form a t-$(n,d,\lambda)_q$ design if $s \leq d-t$?

2 Preliminaries

In this paper we study $\mathbb{F}_{q^m}$-linear codes $C \leq \mathbb{F}_{q^m}^n$ endowed with the rank metric distance. To be more precise, note that the field $\mathbb{F}_{q^m}$ may be viewed as an m-dimensional vector space over $\mathbb{F}_q$. The *rank weight*, or briefly the *weight* of a vector $v=(v_1,\ldots,v_n) \in \mathbb{F}_{q^m}^n$ is defined as the maximum number of coordinates in v that are linearly independent over $\mathbb{F}_q$, i.e., $\mathrm{wt}(v) = \dim_{\mathbb{F}_q}\langle v_1,\ldots,v_n\rangle$. For $v,u \in \mathbb{F}_{q^m}^n$ the rank metric distance is then given by $\mathrm{d}(v,u) = \mathrm{wt}(u-v) = \mathrm{rank}(v-u)$.

A $\mathbb{F}_{q^m}$-linear subspace $C \leq \mathbb{F}_{q^m}^n$ of dimension k endowed with this metric is called a $[n,k]$ $\mathbb{F}_{q^m}$*-linear rank metric* code. As usual the minimum distance of $C \neq \{0\}$ is defined by

$$\mathrm{d} = \mathrm{d}(C) = \min\{\mathrm{wt}(c) \mid 0 \neq c \in C\}.$$

By $A_i(C)$ we always denote the code words of C of weight i. Finally, we use the notation $C^\perp$ for the orthogonal of C which is taken with respect to the standard inner product of $\mathbb{F}_{q^m}^n$.

Throughout the paper we always assume that $C \leq \mathbb{F}_{q^m}^n$ is an $\mathbb{F}_{q^m}$-linear rank metric code with minimum distance d. Furthermore we assume that C is not trivial, i.e., $0 \neq C \neq \mathbb{F}_{q^m}^n$ and $n \leq m$. Thus, if $\dim C = k$, then the last condition implies the Singleton bound

$$d \leq n - k + 1.$$

C is called a *maximum rank distance code*, shortly an MRD code, if the bound is achieved. Delsarte [10] and independently Gabidulin [13] proved the existence of such codes for all q, m, n and dimension $1 \leq k \leq n$ (here $n \leq m$ is not necessary). Given the parameters q, m, n, k, the code $C \leq \mathbb{F}_{q^m}^n$ these authors describe has a particular construction through a special generator matrix and the resulting code is usually called a Gabidulin code. Recently other new constructions of MRD codes have been found which are not equivalent to Gabidulin codes [9, 22]. Somehow surprisingly, over the algebraic closure of F_q, the set of MRD codes forms a generic set inside the Grassmann variety of all k-dimensional linear subspaces of $\mathbb{F}_{q^m}^n$ [20]. In particular over some large finite field there exist large numbers of MRD codes and lower bounds on these cardinalities can be found in [20].

In analogy to the Singleton defect for classical codes as given in [6, 12], we have the following definition for the defect of rank metric codes [8].

Definition 2.1. The *rank defect*, briefly the *defect*, of an $\mathbb{F}_{q^m}$-linear $[n, k, d]$ rank metric code $C \leq \mathbb{F}_{q^m}^n$ is defined by $\mathrm{def}(C) = n - k + 1 - d$.

Note that $\mathrm{def}(C) = 0$ if and only if C is an MRD code. Other interesting codes, which are coming close to MRD codes, are the so-called *dually almost* MRD codes or simply *dually* AMRD codes [7]. More precisely, we say that a $\mathbb{F}_{q^m}$-linear rank metric code C is dually AMRD if $\mathrm{def}(C) = \mathrm{def}(C^\perp) = 1$. Dually AMRD codes are subject of the main results in the last section of this paper. These codes can be viewed as q-analogs of classical almost-MDS (AMDS) codes and as in the classical situation these codes induce again some q-Steiner system.

Let $b_1, \ldots, b_m$ be a basis B of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. For $v = (v_1, \ldots, v_n) \in \mathbb{F}_{q^m}^n$ we write

$$v_i = \sum_{j=1}^{m} \alpha_{ji} b_j$$

and put $M_B(v) = (\alpha_{ji}) \in (\mathbb{F}_q)^{m \times n}$. As mentioned in ([16], Section 2), the $\mathbb{F}_q$-linear row space of $M_B(v)$ is independent of the chosen basis B.

In order to define generalized rank weights we need the following notations [14, 16].

Definition 2.2. For $v = (v_1, \ldots, v_n) \in \mathbb{F}_{q^m}^n$ and an $\mathbb{F}_{q^m}$-linear subspace V of $\mathbb{F}_{q^m}^n$ we define

a) $\mathrm{supp}(v)$ as the $\mathbb{F}_q$-linear row space of $M_B(v)$.

b) $\mathrm{supp}(V) = \langle \mathrm{supp}(v) \mid v \in V \rangle$ as an $\mathbb{F}_q$-vector space.

c) $\mathrm{wt}(V) = \dim \mathrm{supp}(V)$.

d) $V^\star = \sum_{i=0}^{m-1} V^{q^i}$ where $V^{q^i} = \{(v_1{}^{q^i}, \ldots, v_n{}^{q^i}) \mid (v_1, \ldots, v_n) \in V\}$.

In the literature there are different definitions for generalized rank weights (see [21],[19], [11], [16]). All of them define the same numbers. For our purpose the definition given in [16] seems to be the most appropriate.

Definition 2.3. The *r-th generalized rank weight* d_r of a rank metric code $C \leq \mathbb{F}_{q^m}^n$ is defined by

$$\mathrm{d}_r(C) = \min_{\substack{D \leq C \\ \dim D = r}} \mathrm{wt}(D).$$

Combining results of [19],[11] and [16] we obtain the rank metric analog of Wei's result [23] on generalized Hamming weights.

Theorem 2.4. *If C is an $\mathbb{F}_{q^m}$-linear rank metric code in $\mathbb{F}_{q^m}^n$ of dimension k and minimum distance d, then*

$$\mathrm{d}(C) = \mathrm{d}_1(C) < \mathrm{d}_2(C) < \ldots < \mathrm{d}_k(C).$$

Proof. We have

$$\begin{array}{rcll} \mathrm{d}_r(C) & = & \min\limits_{\substack{D \leq C \\ \dim D = r}} \mathrm{wt}(D) & \\ & = & \min\limits_{\substack{D \leq C \\ \dim D = r}} \dim D^\star & ([16], \text{Corollary 4.4}) \\ & = & \min\limits_{\substack{D \leq C \\ \dim D = r}} \max_{d \in D^\star} \mathrm{wt}(d) & ([16], \text{Theorem 5.8}) \\ & = & \min\limits_{\substack{V = V^\star \\ \dim(C \cap V) \geq r}} \dim V & ([11], \text{Proposition II.1}) \\ & = & \mathcal{M}_r(C) & (\text{by Definition 5 in [19]}) \end{array}$$

where

$$\mathcal{M}_r(C) = \min\{\dim V \mid V^q = V \leq \mathbb{F}_{q^m}^n, \dim(C \cap V) \geq r\}.$$

By ([19], Lemma 9) we get

$$\mathcal{M}_1(C) < \ldots < \mathcal{M}_k(C),$$

and the proof is complete since obviously $\mathrm{d}(C) = \mathrm{d}_1(C)$. □

3 Supports of the minimum weight vectors

From [16] we know the following facts.

Lemma 3.1. *Let $C \leq \mathbb{F}_{q^m}^n$ be an $\mathbb{F}_{q^m}$-linear rank metric code.*

a) *If $u = \alpha v$ for some $\alpha \in \mathbb{F}_{q^m}^*$, then* $\mathrm{supp}(v) = \mathrm{supp}(u)$.

b) *If* $v_1, \ldots, v_k \in \mathbb{F}_{q^m}^n$ *generate* C*, then*

$$\operatorname{supp}(C) = \sum_{i=1}^{k} \operatorname{supp}(v_i).$$

c) *There exists an element* $c \in C$ *such that*

$$\operatorname{supp}(c) = \operatorname{supp}(C).$$

d) *For* $u, v \in \mathbb{F}_{q^m}^n$ *there exist* $\alpha, \beta \in \mathbb{F}_{q^m}$ *such that* $\operatorname{supp}(\alpha v + \beta u) = \operatorname{supp}(v) + \operatorname{supp}(u)$.

Proof. a) and b) are part of Proposition 2.3 of [16]. c) is Proposition 3.6 and d) Proposition 3.9 of the same paper. □

Definition 3.2. For an $\mathbb{F}_{q^m}$-linear rank metric code $C \leq \mathbb{F}_{q^m}^n$ of dimension k and minimum distance d we put

$$D_i(C) = \{\operatorname{supp}(c) \mid c \in C,\ \operatorname{wt}(c) = i\}$$

for $i = 0, d, \ldots, n - k + 1$.

Lemma 3.3. *Let* $C \leq \mathbb{F}_{q^m}^n$ *be an* $\mathbb{F}_{q^m}$*-linear rank metric code with minimum distance* d*.*

a) *Let* $v, u \in C$ *and* $\operatorname{wt}(v) = \operatorname{wt}(u) = d$*. Then* $\operatorname{supp}(v) = \operatorname{supp}(u)$ *if and only if there exists* $\alpha \in \mathbb{F}_{q^m}^{\star}$ *such that* $u = \alpha v$*.*

b) $|D_d(C)| = \frac{A_d(C)}{q^m - 1}$.

Proof. a) One direction follows by Lemma 3.1 a). Suppose $\operatorname{supp}(v) = \operatorname{supp}(u)$ and v, u linearly independent over $\mathbb{F}_{q^m}$. Let $W = \langle v, u \rangle$ as a vector space over $\mathbb{F}_{q^m}$. By Lemma 3.1 b), we get $\operatorname{supp}(W) = \operatorname{supp}(v) + \operatorname{supp}(u) = \operatorname{supp}(v)$. Therefore

$$\operatorname{wt}(W) = \dim_{\mathbb{F}_q}(\operatorname{supp}(W)) = \dim_{\mathbb{F}_q}(\operatorname{supp}(v)) = d.$$

Thus, according to the definition of generalized rank weights we obtain

$$d_2(C) = \min\{\operatorname{wt}_R(S) \mid S \leq C \text{ and } \dim_{\mathbb{F}_{q^m}} S = 2\} = d,$$

which contradicts Theorem 2.4.
b) This immediately follows from part a). □

4 q-analog Steiner systems and rank metric codes

Maximum distance separable (MDS) codes are $[n,k,d]$ linear codes $C \leq \mathbb{F}_q^n$ which reach the Singleton bound $d = n - k + 1$. Almost-MDS (AMDS) codes were introduced by de Boer [6] and they are characterized by the Singleton defect one, i.e. $d = n - k$.

In [12] it has been shown that the supports of code words of minimum weight of a $[2d,d,d]$ dually AMDS code ($d \geq 2$) which has no code words of weight $d+1$ form the blocks of an $S(d-1,d,2d)$ classical Steiner system and $d+1$ must be a prime. For instance, in this way the extended ternary Golay code leads to an $S(5,6,12)$ Steiner system. In this section we prove the q-analog of this result.

Definition 4.1. Let q be a prime power and let a and b be non-negative integers. The q-ary Gaussian binomial coefficient of a over b is defined by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{cases} \frac{(q^a-1)(q^{a-1}-1)\cdots(q^{a-b+1}-1)}{(q^b-1)(q^{b-1}-1)\cdots(q-1)} & \text{if } b \leq a \\ 0 & \text{if } b > a \end{cases}$$

Recall that $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is the number of b-dimensional subspaces contained in an a-dimensional $\mathbb{F}_q$-vector space and that the number of blocks of an $S(t,k,n)_q$ Steiner system is $\frac{\begin{bmatrix} n \\ t \end{bmatrix}_q}{\begin{bmatrix} k \\ t \end{bmatrix}_q}$. In the following we freely use the symmetry of the Gaussian binomial coefficients, i.e., $\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{bmatrix} a \\ a-b \end{bmatrix}_q$.

Lemma 4.2. *A $S_q(t,k,n)$ Steiner system implies an $S(t-1,k-1,n-1)_q$ Steiner system if $t \geq 2$.*

Proof. This is one part of ([18], Lemma 5). □

The next Theorem as well Corollary 4.7 may be seen as q-analogs of results in [12].

Theorem 4.3. *Let $C \leq \mathbb{F}_{q^m}^{2d}$ be a $[2d,d,d]$ dually AMRD code with $d \geq 2$ and $A_{d+1}(C) = 0$. Then the set $D_d(C)$ are the blocks of an $S(d-1,d,2d)_q$ Steiner system.*

Proof. (i) Let $W \leq \mathbb{F}_q^{2d}$ be of dimension $d-1$. Suppose that W is contained in two different blocks, i.e., elements of $D_d(C)$. Hence

$$W \leq \operatorname{supp}(u) \cap \operatorname{supp}(v)$$

with $\operatorname{supp}(u), \operatorname{supp}(v) \in D_d(C)$. Since $\dim(\operatorname{supp}(u) \cap \operatorname{supp}(v)) \leq d-1$ we obtain

$$W = \operatorname{supp}(u) \cap \operatorname{supp}(v).$$

Thus

$$\dim(\operatorname{supp}(u) + \operatorname{supp}(v)) = 2d - (d-1) = d+1.$$

By Lemma 3.1 d) there are $\alpha, \beta \in \mathbb{F}_{q^m}$ such that

$$\operatorname{supp}(u) + \operatorname{supp}(v) = \operatorname{supp}(\alpha u + \beta v).$$

Thus $\alpha u + \beta v \in C$ has weight $d+1$, a contradiction. This means that every $(d-1)$-dimensional subspace of $\mathbb{F}_q^{2d}$ is contained in at most one block.
(ii) According to Lemma 3.3 b) we have $|D_d(C)| = \frac{A_d(C)}{q^m-1}$. Since $A_{d+1}(C) = 0$, Theorem 27 of [8] yields

$$A_d(C) = \frac{\begin{bmatrix} 2d \\ d+1 \end{bmatrix}_q}{\begin{bmatrix} d \\ 1 \end{bmatrix}_q}(q^m-1) = \frac{\begin{bmatrix} 2d \\ d-1 \end{bmatrix}_q}{\begin{bmatrix} d \\ d-1 \end{bmatrix}_q}(q^m-1),$$

hence $|D_d(C)| = \frac{\begin{bmatrix} 2d \\ d-1 \end{bmatrix}_q}{\begin{bmatrix} d \\ d-1 \end{bmatrix}_q}$. Since each block contains exactly $\begin{bmatrix} d \\ d-1 \end{bmatrix}_q$ subspaces of dimension $(d-1)$ and every $(d-1)$-dimensional subspace is contained in at most one block by (i), the blocks altogether contain

$$|D_d(C)| \begin{bmatrix} d \\ d-1 \end{bmatrix}_q = \begin{bmatrix} 2d \\ d-1 \end{bmatrix}_q$$

subspaces of dimension $d-1$. As $\begin{bmatrix} 2d \\ d-1 \end{bmatrix}_q$ is the number of $(d-1)$-dimensional subspaces in a space of dimension $2d$, the proof is complete. □

Remark 4.4. Let $C \leq \mathbb{F}_{q^m}^{2d}$ be a $[2d, d, d]$ dually AMRD code with $d \geq 2$ and $A_{d+1}(C) = 0$. Then $C^\perp$ also leads to an $S(d-1, d, 2d)_q$ Steiner system, since C is formally self-dual due to ([7], Lemma 4.11).

Example 4.5. Let C be the $\mathbb{F}_{2^4}$-linear $[4, 2, 2]$ code with generator matrix

$$\begin{pmatrix} 0 & 1 & \omega & 0 \\ 1 & 0 & 0 & \omega \end{pmatrix}$$

where ω is a primitive third root of unity in $\mathbb{F}_{2^4}^*$. With MAGMA [1] we get $A_0(C) = A_0(C^\perp) = 1$, $A_2(C) = A_2(C^\perp) = 75$, $A_3(C) = A_3(C^\perp) = 0$ and $A_4(C) = A_4(C^\perp) = 180$. Thus C is a $[4, 2, 2]$ dually almost MRD code over $\mathbb{F}_{2^4}$. Consequently, by Theorem 4.3 the elements of $D_d(C)$ are the blocks of an $S(1, 2, 4)_2$ Steiner system. Note that this 2-Steiner system is one of the well known spreads.

By Lemma 4.2, the existence of a q-Steiner system $S(d-1, d, 2d)_q$ for $d \geq 2$ implies the existence of $S(1, 2, d+2)_q$. Since the number of blocks of such a Steiner system is $\frac{\begin{bmatrix} d+2 \\ 1 \end{bmatrix}_q}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q}$ it follows that $q^2 - 1 \mid q^{d+2} - 1$. Thus d must be even, hence $d+1$ odd.

Theorem 4.6. *If a q-Steiner system $S(d-1, d, 2d)_q$ with $d \geq 2$ exists, then $d+1$ is a prime.*

Proof. Let p be a prime with $p \mid d+1 \neq p$, hence $d+1 = px$ with $x \geq 2$. Since $p-1 \leq d-1$ Lemma 4.2 implies the existence of an $S(p-1, p, d+p)_q$ Steiner system. This Steiner system has exactly

$$\frac{\begin{bmatrix} d+p \\ p-1 \end{bmatrix}_q}{\begin{bmatrix} p \\ p-1 \end{bmatrix}_q} = \frac{\begin{bmatrix} d+p \\ p-1 \end{bmatrix}_q}{\begin{bmatrix} p \\ 1 \end{bmatrix}_q} \in \mathbb{N}$$

blocks. We have

$$\frac{\begin{bmatrix} d+p \\ p-1 \end{bmatrix}_q}{\begin{bmatrix} p \\ 1 \end{bmatrix}_q} = \frac{(q^{d+p}-1)(q^{d+p-1}-1)\cdots(q^{d+2}-1)}{(q^2-1)(q^3-1)\cdots(q^p-1)}. \qquad (*)$$

Since $d+1 = px$ we see that $p \nmid d+i$ for $i = 2, \ldots, d+p$. Note that p is odd. Thus, by Zsigmondy's Theorem ([15], Chap. IX, Theorem 8.3), there exists a prime r such that $r \mid q^p - 1$ but $r \nmid q-1$. Since $\gcd(q^n-1, q^m-1) = q^{\gcd(m,n)} - 1$ the prime r does not divide any of the factors of the numerator in $(*)$, a contradiction. □

Corollary 4.7. *Let $C \leq \mathbb{F}_{q^m}^{2d}$ be a $[2d, d, d]$ dually AMRD code with $d \geq 2$ and $A_{d+1}(C) = 0$. Then $d+1$ is a prime.*

Proof. This is an immediate consequence of Theorem 4.3 and Theorem 4.6. □

Remark 4.8. In ([12], Theorem 25) it has been shown that a $[2d, d, d]$ dually AMDS code with no code words of weight $d+1$ and $d+2$ is either the binary $[8, 4, 4]$ Hamming code or the ternary $[12, 6, 6]$ Golay code. In contrast, a $[2d, d, d]$ dually AMRD code in $\mathbb{F}_{q^m}^{2d}$ $(2d \leq m)$ with $A_{d+1} = 0 = A_{d+2}$ does not exist. This can be seen as follows. Using the weight distribution of the code ([8], Theorem 27) we get

$$A_d = \frac{\begin{bmatrix} 2d \\ d+1 \end{bmatrix}_q}{\begin{bmatrix} d \\ 1 \end{bmatrix}_q}(q^m - 1)$$

since $A_{d+1} = 0$, and

$$A_d = \frac{\begin{bmatrix} 2d \\ d+2 \end{bmatrix}_q}{q\begin{bmatrix} d \\ 2 \end{bmatrix}_q}(q^m - 1)\left\{ \begin{bmatrix} d+2 \\ 1 \end{bmatrix}_q - (q^m+1) \right\}$$

since $A_{d+2} = 0$. Comparing these equations leads to

$$1 = \frac{q^2-1}{q(q^{d+2}-1)}\left\{ \begin{bmatrix} d+2 \\ 1 \end{bmatrix}_q - (q^m+1) \right\},$$

hence

$$q^{m+2} - q^m + q^2 = q^{d+2},$$

which has no solution since $2 \leq 2d \leq m$.

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References

[1] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system I: The user language, *J. of Symbolic Computation* **24** (1997), 235-265.

[2] M. Braun, T. Etzion, P.R.J. Östergård, A. Vardy and A. Wassermann, Existence of q-analogs of Steiner systems, *Forum Math. Pi* **4** (2016), 14pp., doi:10.1017/fmp.2016.5.

[3] M. Braun, M. Kiermaier, and A. Nakić, On the automorphism group of a binary q-analog of the Fano plane, *European J. of Combinatorics* **51** (2016), 443-457.

[4] T. Bu, Partitions of a vector space, *Discrete Mathematics* **31** (1980), 79-83.

[5] W. Y.C. Chen, Q.-H. Hou, Factors of the Gaussian coefficients, *Discrete Mathematics* **306** (2006), 1446-1449.

[6] M. A. de Boer. Almost MDS codes. *Designs, Codes and Cryptography* **9** (1996), 143-155.

[7] J. de la Cruz, On dually almost MRD codes, `https://arxiv.org/abs/1612.04268`.

[8] J. de la Cruz, E. Gorla, H. López, A. Ravagnani, Weight distribution of rank-metric codes, *Designs, Codes and Cryptography* **86** (2018), 1-16.

[9] J. de la Cruz, M. Kiermaier, A. Wassermann and W. Willems, Algebraic structures of MRD Codes, *Advances in Mathematics of Communications* **10** (2016), 499-510.

[10] P. Delsarte, Bilinear forms over a finite field, with applications to coding theory, *J. of Combinatorial Theory Ser. A* **25** (1978), 226-241.

[11] J. Ducoat, Generalized rank weights: duality and Griesmer bound, `http://arxiv.org/abs/1306.3899v1`

[12] A. Faldum and W. Willems, Codes of small defect, *Designs, Codes and Cryptography***10** (1997), 341-350.

[13] E. M. Gabidulin, Theory of codes with maximum rank distance, *Problems of Information Transmission* **21** (1985), 1-12.

[14] A.-L. Horlemann-Trautmann, K. Marshall and J. Rosenthal, Extension of Overbeck's attack for Gabidulin-based cryptosystems, *Designs, Codes and Cryptography* **86** (2018), 319-340.

[15] B. Huppert and N. Blackburn, Finite Groups II, *Springer Verlag*, Berlin 1982.

[16] R. Jurrius and R. Pellikann, On defining generalized rank weights, *Advances in Mathematics of Communications* **11** (2017), 225-235.

[17] M. Kiermaier, S. Kurz, and A. Wassermann, The order of the automorphism group of a binary q-analog of the Fano plane is at most two, *Designs, Codes and Cryptography* **86** (2018), 239-250.

[18] M. Kiermaier and R. Laue, Derived and residual subspace designs, *Advances in Mathematics of Communications* **9** (2015), 105-110.

[19] J. Kurihara, R. Matsumoto, and T. Uyematsu, Relative generalized rank weight of linear codes and its applications to network coding, *IEEE Transactions on Information Theory* **61** (2015), 3912-3936.

[20] A. Neri, A.-L. Horlemann-Trautmann, T. Randrianarisoa, and J. Rosenthal, On the genericity of maximum rank distance and Gabidulin codes, *Designs, Codes and Cryptography* **86** (2018), 341-363.

[21] F. Oggier and A. Sboui A, On the existence of generalized rank weights, *in Proc. 2012 Int. Symp. Information Theory and Its Applications, Honolulu, Hawaii, USA*, 406-410.

[22] J. Sheekey, A new family of linear maximum rank distance codes, *Advances in Mathematics of Communications* **10** (2016). 475-488.

[23] V. K. Wei, Generalized Hamming weights for linear codes, *IEEE Transactions on Information Theory* **37** (1991), 1412-1418.