ON THE QUADRATIC TYPE OF SOME SIMPLE SELF-DUAL MODULES OVER FIELDS OF CHARACTERISTIC TWO

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Introduction

Let G be a finite group and let K be an algebraically closed field of characteristic 2. Let V be a non-trivial simple self-dual KG-module (we say that V is self-dual if it is isomorphic to its dual V^*). It is a theorem of Fong, [4, Lemma 1], that in this case there is a non-degenerate G-invariant alternating bilinear form, F, say, defined on $V \times V$. We say that V is a KG-module of quadratic type if F is the polarization of a non-degenerate G-invariant quadratic form defined on V. In a previous paper, [7], the present authors have described some methods to decide if such a module V is of quadratic type. One of the main results of [7] is the following. Suppose that G is a group with a split (B, N)-pair of odd characteristic p. Let V be a simple self-dual KG-module that is not of quadratic type and let φ be the Brauer character of V. Then there exists a complex irreducible character of G that occurs as a constituent of the induced character 1_B^G and contains φ as a modular constituent with non-zero multiplicity. This result suggests that we should investigate the decomposition modulo 2 of the irreducible characters in 1_B^G when G is a group of Lie type of odd characteristic and see which real-valued irreducible Brauer characters occur as constituents. However, as far as we know, the decomposition modulo 2 of these characters is not known in complete detail, except in cases when G has small Lie rank or when the relevant constituent of 1_B^G is of special type.

In this paper, we will pursue the investigation suggested above for certain irreducible constituents of 1_B^G which are reasonably accessible and, we hope, important in attempting to understand general principles. The irreducible characters obtained from the doubly transitive permutation actions of the groups of Lie type fall into this category and we give attention to these characters in the first two sections of the paper. We also prove the following existence theorem for simple self-dual modules that are not of quadratic type. Let G be a finite Chevalley group defined over the field $\mathbf{F}_q$. Then if $q \equiv 1 \bmod 4$, there exists a non-trivial simple self-dual module which is not of quadratic type and occurs with multiplicity 1 in the reduction modulo 2 of the Steinberg module of G. There is an analogous result for most of the twisted Chevalley groups. On the other hand, we have not yet found any example of a non-trivial simple self-dual module which is not of quadratic type when G is a (non-twisted) Chevalley group defined over $\mathbf{F}_q$ with $q \equiv 3 \bmod 4$. This raises the question of whether such modules exist in these circumstances. For the group $\mathrm{PSL}(3,q)$, we show that all non-trivial simple self-dual modules are of quadratic type when $q \equiv 3 \bmod 4$, whereas there is exactly one non-trivial simple self-dual module that is not of quadratic type when $q \equiv 1 \bmod 4$. We also classify the non-trivial simple self-dual modules that are not of quadratic type in the case of $\mathrm{PSU}(3,q)$.

In the final section of the paper, we attempt to give some estimate of the number of simple self-dual KG-modules that are not of quadratic type when G is a group of Lie type of odd characteristic. Our method is sufficient to show that this number is bounded by a function of n alone for all q when G is the general linear group $\mathrm{GL}(n,q)$ but our findings are only partial for other groups of Lie type.

1. *Permutation modules in characteristic 2.*

In this section, we investigate whether certain simple self-dual modules that occur as constituents of permutation modules are of quadratic type. Suppose that G acts as a transitive permutation group on a set Ω, where $|\Omega| = n$, and let V be the corresponding permutation module over K with permutation basis $v_1, \ldots, v_n$. We define a G-invariant quadratic form Q on V by

$$Q(\sum_{i=1}^{n} \lambda_i v_i) = \sum_{j<k} \lambda_j \lambda_k.$$

The polarization of Q is the alternating form F given by $F(v_i, v_j) = \delta_{ij} + 1$. It is easy to check that F is non-degenerate if n is even and has rank $n-1$ if n is odd. Let S be the one-dimensional subspace generated by the element $u = v_1 + \cdots + v_n$. S is the subspace of G-fixed points of V. If n is odd, S is complemented by the KG-submodule T defined by

$$T = \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_1 + \cdots + \lambda_n = 0$$

and T is clearly of quadratic type. Suppose now that n is even. $S^\perp$ has codimension 1 in V and $V/S^\perp$ is the trivial KG-module. We call the module $H = S^\perp/S$ the *heart* of the permutation module V. Our next result gives a criterion to determine the quadratic type of simple self-dual constituents of H in certain cases.

THEOREM 1.1. *Using the notation introduced above, suppose that $n \geq 4$ is even and* *each composition factor W_i of the heart H of V is non-trivial, self-dual and occurs with multiplicity* 1. *Then if $n \equiv 0 \bmod 4$, each W_i is of quadratic type. If $n \equiv 2 \bmod 4$ and G either acts primitively on Ω or contains no normal subgroup of index 2, at least one W_i is not of quadratic type.*

Proof. It follows from the definition of the quadratic form Q introduced above that

$$Q(u) = Q(v_1 + \cdots + v_n) \equiv \frac{n(n-1)}{2} \bmod 2.$$

Thus we have $Q(u) = 0$ if $n \equiv 0 \bmod 4$ and $Q(u) = 1$ if $n \equiv 2 \bmod 4$. Consequently, if $n \equiv 0 \bmod 4$, the heart H of V is of quadratic type and we deduce from [7, Lemmas 1.1 and 1.2] that each W_i is of quadratic type in this case. Suppose now that $n \equiv 2 \bmod 4$. As F induces a non-degenerate alternating bilinear form on $H \times H$ and each composition factor of H is self-dual and occurs with multiplicity 1, it follows from the arguments used in the proof of [7, Lemma 1.2] that H is completely reducible, with, say, $H \cong W_1 \oplus \cdots \oplus W_r$ for some positive integer r. Let U_i be the inverse image in $S^\perp$ of W_i for $1 \leq i \leq r$. We claim that there is at least one index j such that S is the socle of U_j. For suppose that this is not the case. Then each U_i is the direct sum of S and a simple submodule isomorphic to W_i. We shall let W_i denote this simple submodule of U_i. It is straightforward to see that we have a direct sum decomposition

$$V = W_1 \oplus \cdots \oplus W_r \oplus X$$

of V into KG-submodules, where X is an extension of S by the trivial module. Let N be the kernel of G acting on X. It is clear that $|G : N|$ is a power of 2. We thus reach a contradiction if G contains no normal subgroup of index 2. On the other hand, if G acts primitively on Ω, N acts transitively on Ω and then S must be the subspace of N-fixed points in V. This contradicts the fact that N acts trivially on X. We deduce that S is the socle of some U_j. Finally, as Q clearly induces a non-degenerate quadratic form on each U_i, it follows from [7, Theorem 1.3] that W_j is not of quadratic type. ■

We would like to make two remarks concerning Theorem 1.1. Firstly, the conclusion of Theorem 1.1 in the case that $n \equiv 2 \bmod 4$ may be incorrect if G acts imprimitively and contains a subgroup of index 2. An example is provided by the wreath product $S_3 \wr S_2$

acting on six points. The heart H of the corresponding permutation module is simple and of dimension 4 over $\mathbf{F}_2$. Moreover, H is of quadratic type, a fact which proves that the orthogonal group $O^+(4,2)$ is isomorphic to $S_3 \wr S_2$. Secondly, suppose we know that S is the socle of V. Then if $n \equiv 2 \bmod 4$, none of the simple modules W_i is of quadratic type. For, if we follow the proof of the Theorem, S must be the socle of each submodule U_i and we deduce, as above, that W_i is not of quadratic type. No further assumptions concerning the structure or action of G are necessary. We will see that this situation occurs in Section 2, when we discuss the unitary group $\mathrm{PSU}(3,q)$.

The simplest examples that we know of groups that satisfy the hypotheses of Theorem 1.1 occur among the doubly transitive permutation groups. We will consider a number of cases of such groups here. We begin with the symmetric group S_n. If n is odd, the complement T of S that we introduced above is certainly of quadratic type and moreover is simple, as observed in [15, Section 3 (A)]. When n is even, the heart H of V is also simple and by Theorem 1.1 is of quadratic type if and only if $n \equiv 0 \bmod 4$. This fact is recorded in [12, p.187].

We consider next the projective special linear group $\mathrm{PSL}(n,q)$, where q is a power of an odd prime and $n \geq 2$. $\mathrm{PSL}(n,q)$ is a doubly transitive group of degree $(q^n-1)/(q-1)$. We note that $(q^n-1)/(q-1)$ is odd if and only if n is odd. Suppose first that n is odd. Using the notation introduced above, Mortimer shows in [15, Section 3 (C)] that T is simple and we have observed that T is of quadratic type. Our next theorem describes what happens when n is even.

THEOREM 1.2. *Let $n \geq 2$ be an even integer and let q be a power of an odd prime. Let H be the heart of the permutation module over K obtained from the doubly transitive action of* $\mathrm{PSL}(n,q)$ *of degree $(q^n-1)/(q-1)$.*

(a) *Suppose that $n \geq 4$. Then H is simple and it is of quadratic type if and only if either $n \equiv 0 \bmod 4$ or $n \equiv 2 \bmod 4$ and $q \equiv 3 \bmod 4$.*

(b) *Suppose that $n = 2$. Then H is the sum of two non-isomorphic simple modules of dimension $(q-1)/2$. These modules are not self-dual if $q \equiv 3 \bmod 4$. If $q \equiv 1 \bmod 4$, they* *are self-dual but not of quadratic type.*

Proof. The statements about the simplicity of H follow from [15]. Furthermore, as it is easy to see that $(q^n - 1)/(q - 1) \equiv 0 \bmod 4$ if and only if either $n \equiv 0 \bmod 4$ or $n \equiv 2 \bmod 4$ and $q \equiv 3 \bmod 4$, part (a) of the theorem follows from Theorem 1.1. The statements about the self-duality of the simple constituents of H in part (b) follow from a consideration of the associated Brauer characters. In the case that $q \equiv 1 \bmod 4$ and $n = 2$, Theorem 1.1 shows that at least one simple constituent of H is not of quadratic type. However, the two simple constituents are well known to be conjugate under an automorphism of $\mathrm{PSL}(2, q)$ induced by conjugation by a suitable element of $\mathrm{PGL}(2, q)$ and it follows that if one module is not of quadratic type, neither is the other. ■

Analysis of the hearts of the permutation modules of other doubly transitive groups is generally more complicated than the case described in Theorem 1.2. In the next section of this paper, we investigate the doubly transitive action of $\mathrm{PSU}(3, q)$ on q^3+1 points. In a previous paper, [8], the authors investigated the quadratic type of the simple constituents of the heart of the permutation modules of a Ree group acting in a doubly transitive manner. The simple constituents of the hearts of the permutation modules obtained from the two doubly transitive actions of $\mathrm{Sp}(2n, 2)$ are related to those obtained from the exterior powers of the natural $2n$-dimensional module for $\mathrm{Sp}(2n, 2)$ over $\mathbf{F}_2$ and are not suitable for study by the methods described in this paper.

2. *The quadratic type of simple self-dual modules of* $\mathrm{PSU}(3, q)$.

Let q be a power of an odd prime p. The projective special unitary group $\mathrm{PSU}(3, q)$ has a doubly transitive permutation action on $q^3 + 1$ points. The stabilizer of a point in this action is a Borel subgroup B of $\mathrm{PSU}(3, q)$. We set $G = \mathrm{PSU}(3, q)$ and let St denote the non-trivial irreducible constituent of the permutation character 1_B^G. [7, Theorem 2.3] shows that the Brauer character of any non-trivial simple self-dual module of G which is not of quadratic type occurs as a 2-modular constituent of St with non-zero multiplicity. Using work of Erdmann, [3], giving the 2-modular decomposition of St when $q \equiv 1 \bmod 4$, we now show that there are precisely two non-trivial simple self-dual modules of $\mathrm{PSU}(3, q)$ which are not of quadratic type in this case.

THEOREM 2.1. *Suppose that q is a power of a prime and K is an algebraically closed field of characteristic 2. Then if $q \equiv 1 \bmod 4$, there are precisely two non-trivial simple self-dual* $\mathrm{PSU}(3,q)$*-modules over K that are not of quadratic type. These are the simple constituents of the heart of the doubly transitive permutation module for* $\mathrm{PSU}(3,q)$ *over K and their dimensions are $q(q-1)$ and $(q^2+1)(q-1)$.*

Proof. Let V denote the permutation module over K obtained from the doubly transitive action of $\mathrm{PSU}(3,q)$ of degree q^3+1. Since $q \equiv 1 \bmod 4$, we have $\dim V \equiv 2 \bmod 4$. By [3, Lemma 4.2], the heart of V is the direct sum of two simple self-dual modules, which, to avoid confusion with our earlier notation, we will denote by E_1 and E_2 (Erdmann denotes the modules by E and S). We may assume the notation is chosen so that

$$\dim E_1 = q(q-1), \quad \dim E_2 = (q^2+1)(q-1).$$

Erdmann also shows that the socle of V is the trivial K-module. It follows from the second remark following the proof of Theorem 1.1 that neither E_1 nor E_2 is of quadratic type. Furthermore, as we noted in the introduction to this section, E_1 and E_2 are the unique non-trivial self-dual modules that are not of quadratic type. ■

We assume for the rest of this section that $q \equiv 3 \bmod 4$. The irreducible Brauer characters that occur as 2-modular constituents of St are in the principal 2-block of $\mathrm{PSU}(3,q)$, since the trivial Brauer character is such a constituent, its multiplicity being 1. According to [14, Lemma 3.2.1], the principal 2-block contains exactly three irreducible Brauer characters, these being the trivial Brauer character, a Brauer character φ, say, of degree $q(q-1)$, and a third Brauer character, which we will denote by ψ. Both φ and ψ are real-valued and therefore correspond to non-trivial simple self-dual modules. At the time of this writing, we do not know the multiplicity with which φ occurs as a 2-modular constituent of St, although it is even and at least 2. [14, Theorem 3.2.5] shows that ψ occurs with multiplicity 1 as a 2-modular constituent of St, but the degree of ψ is unknown on account of the multiplicity above being unknown. Now the dimension of the permutation module over K obtained from the doubly transitive action of $\mathrm{PSU}(3,q)$ is divisible by 4, since 4 divides $q^3+1=(q+1)(q^2-q+1)$ in this case. It follows from [7, Lemma 1.2] and the proof of Theorem 1.1 that the simple self-dual module corresponding to ψ

is of quadratic type. The remainder of this section is devoted to proving that the simple self-dual module corresponding to φ is not of quadratic type.

It is shown in [14, Section 3.2] that φ is the restriction to 2-regular elements of an irreducible complex character $\chi_{q(q-1)}$ of degree $q(q-1)$. Reference to the character table of $\mathrm{PSU}(3,q)$ given in [17] indicates that $\chi_{q(q-1)}$ is the unique irreducible character of this degree for the group and no other non-trivial irreducible character has smaller degree. Now $\mathrm{PSU}(3,q)$ is a normal subgroup of index $\gcd(3,q+1)$ in the projective unitary group $\mathrm{PU}(3,q)$ and we intend to show that $\chi_{q(q-1)}$ is obtained as the restriction of a rational-valued complex character of this larger group. To do this, we make use of the results of [6].

The Weil representation of the unitary group $\mathrm{U}(3,q)$ is a complex representation of degree q^3 with rational-valued character that contains exactly $q+1$ different irreducible constituents, each occurring with multiplicity 1. There is a unique irreducible constituent of the Weil representation which is trivial on the centre of $\mathrm{U}(3,q)$ and it has degree $q(q-1)$. See [6, Corollary 4.5]. The character, χ, say, of this constituent must also be rational-valued, on account of its uniqueness. Now as the centre of $\mathrm{U}(3,q)$ is the kernel of χ, we may consider χ to be the character of a faithful irreducible representation of $\mathrm{PU}(3,q)$. It follows from our considerations above that the restriction of χ to $\mathrm{PSU}(3,q)$ must be $\chi_{q(q-1)}$. Furthermore, since $\chi_{q(q-1)}$ defines an irreducible Brauer character modulo 2, the same must be true of χ. Thus there is a simple module, W, say, for $\mathrm{PU}(3,q)$ whose Brauer character is χ restricted to 2-regular elements. W is self-dual, as χ is real-valued, and its restriction to $\mathrm{PSU}(3,q)$ is simple. It is important for us to know that W is defined over $\mathbf{F}_2$. This follows from [11, Theorem 9.14], since W is absolutely irreducible and the modular character of W (as opposed to its Brauer character) takes its values in $\mathbf{F}_2$. (The modular character of W is simply obtained by reduction modulo 2 of the values of χ.) We intend to show that W is not of quadratic type.

We set H equal to $\mathrm{PU}(3,q)$. We note that H contains a cyclic subgroup, C, say, of order q^2-q+1, this being the image of a cyclic subgroup of order q^3+1 in $\mathrm{U}(3,q)$. Consideration of the values of χ shows that C acts without non-zero fixed points on W.

Indeed, as an $\mathbf{F}_2C$-module, W is isomorphic to the augmentation ideal of the group ring $\mathbf{F}_2C$. Furthermore, if Z denotes the centre of a Sylow p-subgroup of H, Z is an elementary abelian p-group of order q which also acts without non-zero fixed points on W. This follows from the fact that χ takes the constant value $-q$ on the non-identity elements of Z. We will argue that W cannot be an $\mathbf{F}_2H$-module of quadratic type by showing that the Witt type of W considered as an $\mathbf{F}_2C$-module is different from the Witt type of W considered as an $\mathbf{F}_2Z$-module.

Suppose by way of contradiction that W is an $\mathbf{F}_2H$-module of quadratic type and let Q be a non-degenerate H-invariant quadratic form defined on W. We set $\varepsilon(Q)$ equal to 1 if the Witt index of Q is maximal and $\varepsilon(Q)$ equal to -1 if the Witt index of Q is less than maximal.

Lemma 2.2. *Assume the notation introduced above. Let m be the order of 2 modulo p. Then $\varepsilon(Q) = 1$ if and only if m divides $q(q-1)/2$.*

Proof. Since Z is an elementary abelian group that acts without non-zero fixed points on W, the result follows from [18, Theorem 5.8]. ∎

The calculation of $\varepsilon(Q)$ based on consideration of the action of C is a little more complicated. Let σ be a generator of C. We observed that W is isomorphic as an $\mathbf{F}_2C$-module to the augmentation ideal of the group ring $\mathbf{F}_2C$ and it follows that the characteristic polynomial f, say, of σ acting on W is $(x^{q^2-q+1}-1)/(x-1)$. [1, Theorem 1.2] gives a means of calculating $\varepsilon(Q)$ in terms of the Berlekamp discriminant $D(f)$ of f. Interpreting [1, Theorem 1.2] in the context of $\mathbf{F}_2$, we see that $\varepsilon(Q) = 1$ if and only if $D(f) = 0$. It is easier to find $D(f)$ by using Swan's method, described in [19], rather than by using Berlekamp's formula. In Swan's method, we lift f to a polynomial F in $\mathbf{Z}[x]$ whose reduction modulo 2 is f and calculate the usual discriminant $\Delta(F)$ of F as the square of the difference product of the roots of F. [19, Corollary 3] implies that $D(f) = 0$ if and only if $\Delta(F) \equiv 1 \bmod 8$. We may take the polynomial F to be $(x^{q^2-q+1}-1)/(x-1)$ in $\mathbf{Z}[x]$. We note without proof how to calculate the discriminant of F.

Lemma 2.3. *Let n be an odd integer. Then the discriminant of $(x^n-1)/(x-1)$ in*

$\mathbf{Z}[x]$ *is* $(-1)^{(n-1)/2}n^{n-2}$.

We now proceed to the proof that W is not of quadratic type.

THEOREM 2.4. *Let* $H = PU(3, q)$, *where* $q \equiv 3 \bmod 4$. *Then if* W *is the simple self-dual* $\mathbf{F}_2H$*-module whose Brauer character is the restriction to 2-regular elements of the unique irreducible complex character of degree* $q(q-1)$ *that is a constituent of the Weil character,* W *is not of quadratic type.*

Proof. Write $q = p^a$, where p is a prime and a is a positive integer. Then as $q \equiv 3 \bmod 4$, a must be odd and p must satisfy $p \equiv 3 \bmod 4$. Suppose that W is of quadratic type and let Q be a non-degenerate H-invariant quadratic form defined on V. Let m be the order of 2 modulo p. As m is a divisor of $p-1$ and $\gcd(p-1,\ q(q-1)/2) = (p-1)/2$ in our case, Lemma 2.2 shows that $\varepsilon(Q) = 1$ if and only if m divides $(p-1)/2$. On the other hand, as the multiplicative group of $\mathbf{F}_p$ is cyclic, m is a divisor of $(p-1)/2$ if and only if 2 is a quadratic residue modulo p. Using the known quadratic character of 2 modulo p, we see that $\varepsilon(Q) = 1$ if $p \equiv 7 \bmod 8$ and $\varepsilon(Q) = -1$ if $p \equiv 3 \bmod 8$.

Now we calculate $\varepsilon(Q)$ by considering the action of the cyclic subgroup C. The criteria of Baeza and Swan discussed above, together with Lemma 2.3, show that

$$\varepsilon(Q) = 1 \text{ if and only if } (-1)^{(q^2-q)/2}(q^2-q+1)^{(q^2-q-1)} \equiv 1 \bmod 8.$$

If $p \equiv 7 \bmod 8$, it follows that $q \equiv 7 \bmod 8$ also and the criterion above yields that $\varepsilon(Q) = -1$ in this case. Similarly, the supposition that $p \equiv 3 \bmod 8$ leads to the conclusion that $\varepsilon(Q) = 1$. As these two results are in conflict with those obtained in the paragraph above, we see that W cannot be of quadratic type. ■

Finally, we show that the restriction of the module W to the normal subgroup $\mathrm{PSU}(3, q)$ of $\mathrm{PU}(3, q)$ is also not of quadratic type, considered as a module for this smaller group. Our proof is based on the following general proposition.

LEMMA 2.5. *Let* G *be a non-trivial normal subgroup of a group* H *and let* L *be a field of characteristic* 2. *Let* M *be a simple self-dual* LH*-module on which* G *acts non-trivially and let* $F : M \times M \to L$ *be an* H*-invariant alternating bilinear form. Suppose that* F *is* *not the polarization of an H-invariant quadratic form defined on M. Then F is also not the polarization of a G-invariant quadratic form defined on M.*

Proof. Suppose that F is the polarization of a G-invariant quadratic form, Q, say. We will show that Q must also be H-invariant. To do this, we define for each element h of H a function $f^h : M \to L$ by

$$f^h(u) = Q(u) + Q(hu)$$

for all $u \in M$. It is easy to check that

$$f^h(\lambda u + \mu v) = \lambda^2 f^h(u) + \mu^2 f^h(v)$$

for all u and v in M and all λ and μ in L. Thus f^h is a semilinear form on M. Furthermore, for all g in G and u in M, we have $f^h(u) = f^h(gu)$, since Q is G-invariant and G is normal in H, and thus

$$f^h(u + gu) = f^h(u) + f^h(gu) = 0.$$

This shows that $u + gu$ is in $\ker f^h$, which is an LG-submodule of M, and thus $M/\ker f^h$ is a trivial LG-module. However, since M is a simple LH-module and G is normal in H, Clifford's theorem implies that M is completely reducible as an LG-module, and thus $\ker f^h$ has a G-invariant complement, which must be a trivial module for G. A second application of Clifford's theorem shows that M must be a trivial LG-module, since it contains a non-zero trivial LG-submodule. This contradicts our hypothesis and thus Q is invariant under h. But now we have proved our claim that Q is also H-invariant and we have again reached a contradiction. Consequently, Q does not exist and our lemma is thus proved. ■

The conclusion of Theorem 2.4, Lemma 2.5 and our discussion after the proof of Theorem 2.1 is the following result.

THEOREM 2.6. *Suppose that q is a power of an odd prime and K is an algebraically closed field of characteristic 2. Then if $q \equiv 3 \bmod 4$ there is a unique non-trivial simple self-dual* $\mathrm{PSU}(3, q)$*-module over K that is not of quadratic type. This module has dimension $q(q-1)$ and it is obtained by reduction modulo 2 of the unique simple module over* $\mathbf{C}$ *of this dimension.*

3. *On the decomposition modulo 2 of the Steinberg module for groups of Lie type of odd characteristic.*

Our investigations of the 2-modular decomposition of the Steinberg characters of $\mathrm{PSL}(2,q)$ and $\mathrm{PSU}(3,q)$ may be used to obtain some information about the 2-modular decomposition of the Steinberg character of a group of Lie type, as we intend to demonstrate in this section. Let G be a finite group of Lie type of characteristic p, where p is an odd prime. We will say that G is defined over the field $\mathbf{F}_q$ if G is either a Chevalley group over $\mathbf{F}_q$ or G is one of the twisted groups ${}^2A_l(q)$, ${}^2D_l(q)$, ${}^2E_6(q)$ or ${}^3D_4(q)$ obtained as the group of fixed points of an automorphism of a Chevalley group of the appropriate type defined over $\mathbf{F}_{q^2}$ or $\mathbf{F}_{q^3}$. We omit the Ree groups of type G_2 from our considerations, as these have already been considered in [8]. Let St_G denote the Steinberg character of G. As p is odd, it follows from a result of Hiss, [9], that St_G contains the trivial Brauer character with multiplicity 1 as a 2-modular constituent. Our intention is now to prove the following theorem.

THEOREM 3.1. *Suppose that q is a power of an odd prime p and K is an algebraically closed field of characteristic 2. Let G be a finite group of Lie type defined over $\mathbf{F}_q$ (as defined above). Suppose also that $q \equiv 1 \bmod 4$. Then there exists a non-trivial simple self-dual KG-module that is not of quadratic type whose Brauer character occurs with multiplicity 1 in the reduction modulo 2 of the Steinberg character St_G of G.*

Proof. Let B be a Borel subgroup of G and let P be a parabolic subgroup of G minimal with respect to containing B as a proper subgroup. Let L be the Levi factor of P. Apart from the case that G is of type ${}^2A_2$, we may choose P so that $\mathrm{PSL}(2,q)$, $\mathrm{PSL}(2,q^2)$, or $\mathrm{PSL}(2,q^3)$ is the only non-abelian composition factor of L. In the exceptional case, we may take G to be $\mathrm{PSU}(3,q)$ and then our theorem follows from Theorem 2.1. Thus, we may assume in all cases that the non-abelian composition factor of L is of type A_1.

Let St_L denote the Steinberg character of L. The restriction of St_G to P will be denoted by $(St_G)|_P$. Let $\overline{St_G}$ denote the reduction of St_G modulo 2, with a corresponding notation for St_L. We have

$$(St_G)|_P = (St_L)^P$$

by [2, Proposition 6.3.3]. Reducing modulo 2, we also have

$$(\overline{St_G})|_P = (\overline{St_L})^P.$$

Furthermore, as P is the semi-direct product of L with its unipotent radical, U, say, consideration of the induced character $(St_L)^P$ shows that the inflation of St_L to P is the only irreducible constituent of $(St_L)^P$ that contains U in its kernel. Then since U has odd order, we see that the inflation of $\overline{St_L}$ to P is also the only constituent of $(\overline{St_L})^P$ that contains U in its kernel. We note also that it straightforward to see that $(\overline{St_G})|_P$ contains the trivial Brauer character of P with multiplicity 1.

As St_L contains the centre of L in its kernel, we may assume that L is of adjoint type and hence that $\mathrm{PSL}(2, q^a) \leq L \leq \mathrm{PGL}(2, q^a)$, where $a = 1$, 2 or 3. Given our assumption that $q \equiv 1 \bmod 4$, Theorem 1.2 now implies that $\overline{St_L}$ contains a non-trivial irreducible Brauer character, φ, say, with multiplicity 1, corresponding to a non-trivial simple self-dual KL-module that is not of quadratic type. Our discussion in the paragraph above shows then that the inflation $\inf(\varphi)$ of φ to P occurs with multiplicity 1 in $(\overline{St_L})^P = (\overline{St_G})|_P$. Finally, we argue that, since $\overline{St_G}$ and $(\overline{St_G})|_P$ each contain the appropriate trivial Brauer character exactly once, there must be a simple constituent, V, say, of the reduction modulo 2 of the Steinberg module whose restriction to P contains a simple self-dual KP-module, W, say, whose Brauer character is $\inf(\varphi)$, and does not contain the trivial KP-module. W cannot be of quadratic type, and since $\inf(\varphi)$ occurs with multiplicity 1 in $(\overline{St_G})|_P$, we readily deduce that V must also be self-dual and must occur with multiplicity 1 in the reduction of the Steinberg module. V also cannot be of quadratic type, since, if it were, the fact that its restriction to P does not contain the trivial KP-module would imply that W was of quadratic type, by [7, Lemma 1.2]. ■

While the supposition that $q \equiv 1 \bmod 4$ seems to be essential for the truth of Theorem 3.1 in the case that G is a Chevalley group, it is not essential when G is one of the twisted types ${}^2A_l(q)$ for $l \geq 3$, ${}^2D_l(q)$ or ${}^2E_6(q)$. This is because, following the notation of the proof of Theorem 3.1, the simple composition factor of the Levi factor L may be taken to be $\mathrm{PSL}(2, q^2)$ and therefore $\overline{St_L}$ always contains a non-trivial irreducible Brauer character with multiplicity 1, corresponding to a non-trivial simple self-dual KL-

module that is not of quadratic type. Thus, when G is one of the twisted groups of the type described above, there exists a non-trivial simple self-dual KG-module that is not of quadratic type whose Brauer character occurs with multiplicity 1 in the reduction modulo 2 of the Steinberg character of G. The group ${}^2A_2(q)$ is a genuine exception here, as no such Levi factor exists, and we have already seen in Section 2 that any Brauer character of the type described above occurs with even multiplicity in the reduction of the Steinberg character of ${}^2A_2(q)$ $(\mathrm{PSU}(3,q))$. We sum up this discussion in our next result.

THEOREM 3.2. *Suppose that q is a power of an odd prime and K is an algebraically closed field of characteristic 2. Let G be a group of type ${}^2A_l(q)$ for $l \geq 3$, ${}^2D_l(q)$ or ${}^2E_6(q)$. Then there exists a non-trivial simple self-dual KG-module that is not of quadratic type whose Brauer character occurs with multiplicity 1 in the reduction modulo 2 of the Steinberg character St_G of G.*

4. *The quadratic type of simple self-dual modules of* $\mathrm{PSL}(3,q)$.

In this section, we assume that q is a power of an odd prime p. We intend to prove that if $q \equiv 3 \bmod 4$, each non-trivial simple self-dual $\mathrm{PSL}(3,q)$-module over K is of quadratic type, whereas if $q \equiv 1 \bmod 4$, there is a unique non-trivial simple self-dual $\mathrm{PSL}(3,q)$-module that is not of quadratic type and it has degree q^3-1. Before proving this statement, we make some preliminary observations.

Set $G = \mathrm{PSL}(3,q)$ and let B denote a Borel subgroup of G. Suppose that ϕ is the Brauer character of some non-trivial simple self-dual G-module that is not of quadratic type. Then by [7, Theorem 2.3], there is an irreducible complex character in 1_B^G that contains ϕ as a 2-modular constituent with non-zero multiplicity. Now there are precisely two non-trivial irreducible constituents of 1_B^G, namely, the character of the doubly transitive representation of G, χ, say, and the Steinberg character St. We know from the discussion before Theorem 1.2 that χ is irreducible modulo 2 and the corresponding simple self-dual module is of quadratic type. A result of Erdmann, [3, Lemma 3.2], implies that St decomposes modulo 2 into the trivial Brauer character and an irreducible Brauer character of degree q^3-1 when $q \equiv 3 \bmod 4$. The same result holds true when $q \equiv 1 \bmod 4$. The key point of the proof is to show that, if A is the stabilizer of a point in the doubly transitive permutation action of G, with $|G : A| = q^2 + q + 1$, then A has an irreducible representation S_0 over K of degree $q - 1$ (obtained from the modular reduction of the doubly transitive permutation representation of a Levi complement of A) such that S_0^G is irreducible of degree $q^3 - 1$. It is the Brauer character of S_0^G that is the non-trivial 2-modular constituent of St. We therefore state, without proof, the following result.

Lemma 4.1. *Suppose that q is a power of an odd prime p. Let St be the Steinberg character of* $\mathrm{PSL}(3, q)$. *Then St decomposes modulo 2 into the trivial Brauer character and an irreducible Brauer character of degree $q^3 - 1$.*

We will call the simple module of dimension $q^3 - 1$ whose existence is implied by Lemma 4.1 the *deleted Steinberg module.*

Theorem 4.2. *Suppose that q is a power of an odd prime p and K is an algebraically closed field of characteristic 2. Then if $q \equiv 3 \bmod 4$, any non-trivial simple self-dual* $\mathrm{PSL}(3, q)$*-module over K is of quadratic type. If $q \equiv 1 \bmod 4$, there is a unique non-trivial simple self-dual* $\mathrm{PSL}(3, q)$*-module over K that is not of quadratic type and it is the deleted Steinberg module.*

Proof. We have noted that any non-trivial simple self-dual $\mathrm{PSL}(3, q)$-module over K that is not of quadratic type is a 2-modular constituent of the Steinberg module. In view of Lemma 4.1 and Theorem 3.1, we see that the deleted Steinberg module is not of quadratic type when $q \equiv 1 \bmod 4$, and this is the unique non-trivial simple self-dual module that is not of quadratic type.

Suppose then that $q \equiv 3 \bmod 4$. Set $G = \mathrm{PSL}(3, q)$ and let B be a Borel subgroup of G. We have

$$|G : B| = (q^2 + q + 1)(q + 1)$$

and thus $|G : B|$ is divisible by 4 in this case. The proof of Theorem 1.1 implies that the heart of the permutation module over K of G acting on the cosets of B is a KG-module of quadratic type. Furthermore, the trivial KG-module is not a constituent of the heart, as its composition factors are the deleted Steinberg module and two copies of the simple module of dimension $q(q + 1)$ obtained from the doubly transitive permutation action of

G. It follows from [7, Lemma 1.2] that the deleted Steinberg module is of quadratic type and thus our theorem holds when $q \equiv 3 \bmod 4$. ■

5. *Some estimates of the number of simple self-dual modules that are not of quadratic type.*

Our findings in Sections 2 and 4 of this paper suggest that if L is a simple group of Lie type of odd characteristic, the number of simple self-dual KL-modules that are not of quadratic type is small compared with the total number of simple self-dual KL-modules and is perhaps bounded by a function dependent on the rank of L, but independent of the size q of the field over which L is defined. Our purpose in this section is to try to prove the existence of such a bound. While we are successful only for one type of group, we hope that our methods may admit improvements and have interest in their own right.

Let G be a simple algebraic group of adjoint type over an algebraically closed field of prime characteristic p and let l be the rank of G. Let $F : G \to G$ be a Frobenius map and let G^F be the corresponding finite group of F-fixed points. There corresponds a real number $q > 1$ (usually a positive integer power of p) and the possibilities for G^F are described in [2, p.39]. Let B be an F-stable Borel subgroup of G and T an F-stable maximal torus of G contained in B. Let U be the unipotent radical of B. U is F-stable and is generated by the root subgroups X_α, where α runs over the positive roots. The X_α are permuted among themselves by F and we obtain in this way a permutation ρ, say, on the set of positive roots which is known to map the set of simple positive roots onto itself. Thus ρ induces a permutation of the simple positive roots, as described in [2, p.34]. Let U^* be the subgroup of U generated by the subgroups X_β, where β runs over the positive non-simple roots. It is known that U^* is a normal F-stable subgroup of U and U/U^* is abelian. See, for example, [2, p.76]. U^F is a Sylow p-subgroup of G^F and $U^F/(U^*)^F$ is an elementary abelian p-group of order q^l, as shown in [2, p.132]. We will make the assumption that $(U^*)^F$ is the derived group of U^F. This is guaranteed if p is a good prime for G, as shown in [13, p.258]. When G is of classical type, this only excludes the prime $p = 2$, which is of no interest in our work in this paper. If G^F is a Chevalley group, which is the case if the permutation ρ is the identity, the only odd value of q excluded is $q = 3$

when G is of type G_2, according to [10, Lemma 7]. Thus, assuming that p is odd, the only prime for which our assumption is not valid is $p = 3$ and this occurs when G^F is either $G_2(3)$ or ${}^2E_6(q)$ with q a power of 3.

The group U^F is normal in B^F, which is a semi-direct product $U^F T^F$, and T^F is an abelian group that normalizes U^F. We wish to study the irreducible complex characters χ of B^F with the property that the irreducible constituents of $\chi|_{U^F}$ are all linear. These may be identified with the irreducible characters of B^F of degree relatively prime to p. Now T^F acts by conjugation on the quotient group $U^F/(U^*)^F$, as described in [2, pp.76-77], and and on the linear characters of this group. The following result concerning the action of T^F on the linear characters is easily deduced from the discussion given in [2, Proposition 8.1.2] and we therefore omit a formal proof.

Lemma 5.1. *In the notation introduced above, suppose that ρ has exactly t orbits on the set of simple positive roots. Then T^F has exactly 2^t orbits in its action on the linear characters of $U^F/(U^*)^F$.*

Let λ be a complex linear character of $U^F/(U^*)^F$. We may consider λ to be a linear character of U^F by inflation. Let $I(\lambda)$ denote the stabilizer of λ in T^F. The stabilizer of λ in B^F is then the semi-direct product $U^F I(\lambda)$. Given any linear character μ of $I(\lambda)$, we may extend λ to a character $\hat{\lambda}\mu$ of $U^F I(\lambda)$ by setting

$$\hat{\lambda}\mu(uh) = \lambda(u)\mu(h)$$

for all $u \in U^F$ and $h \in I(\lambda)$. The induced character $(\hat{\lambda}\mu)^{B^F}$ of B^F is irreducible and different choices of μ lead to different induced characters. Thus, if we select a linear character λ_i of U^F in the i-th T^F-orbit, we obtain $|I(\lambda_i)|$ different irreducible complex characters of B^F and the restriction to U^F of each such character consists of the sum of all the linear characters in the i-th orbit. Any irreducible complex character of B^F whose restriction to U^F contains a linear character of U^F has the form $(\hat{\lambda}_i\mu)^{B^F}$ for a suitable choice of i and μ. When we take μ to be the trivial character of $I(\lambda)$, we will write $\hat{\lambda}$ for the corresponding extension of λ to $U^F I(\lambda)$.

The following result will prove useful when investigating the modular decomposition of the irreducible constituents of $1_{B^F}^{G^F}$.

Lemma 5.2. *Let λ be a linear character of U^F, μ a linear character of $I(\lambda)$ and $(\hat{\lambda}\mu)^{B^F}$ the corresponding irreducible character of B^F. Let π denote the induced character $1_{B^F}^{G^F}$ of G^F and suppose that*

$$(\pi|_{B^F}, (\hat{\lambda}\mu)^{B^F}) \neq 0.$$

Then μ is trivial and

$$(\pi|_{B^F}, (\hat{\lambda}\mu)^{B^F}) \leq |W^F|,$$

where W^F is the Weyl group of G^F. In particular, the multiplicity with which λ appears as a constituent of $\pi|_{U^F}$ is at most $|W^F|$.

Proof. The Bruhat decomposition in G^F shows that there is a double coset decomposition

$$G^F = \bigcup_{w \in W^F} B^F n_w B^F,$$

where the double coset representatives n_w are parametrized by the elements w of the Weyl group W^F of G^F, and each element n_w normalizes T^F. Let 1_w denote the trivial character of the subgroup $n_w B^F n_w^{-1} \cap B^F$. Mackey's subgroup theorem shows that $\pi|_{B^F}$ is the sum of the $|W^F|$ characters of B^F of the form $1_w^{B^F}$. Thus it suffices to show that

$$(1_w^{B^F}, (\hat{\lambda}\mu)^{B^F}) = 0$$

if μ is not the trivial character of $I(\lambda)$ and

$$(1_w^{B^F}, (\hat{\lambda})^{B^F}) \leq 1.$$

For notational convenience, we set $C = U^F I(\lambda)$ and $D = n_w B^F n_w^{-1} \cap B^F$. Now since T^F is contained in D, we have $B^F = CD$. Thus, Mackey's subgroup theorem and Frobenius reciprocity imply that

$$(1_w^{B^F}, (\hat{\lambda}\mu)^{B^F}) = (1_{C\cap D}, \hat{\lambda}\mu|_{C\cap D}).$$

However, as $I(\lambda)$ is contained in $C \cap D$ and μ is an irreducible character of $I(\lambda)$, it is clear that the latter inner product is 0 if μ is non-trivial and either 0 or 1 if μ is trivial. ■

Suppose now that r is a prime different from p. Let $\mathrm{IBr}(G^F, r)$ denote the set of irreducible Brauer characters of G^F modulo r. The corresponding set $\mathrm{IBr}(U^F, r)$ may be identified with the set of irreducible complex characters of U^F, since r does not divide $|U^F|$. We define a subset $\mathrm{IBr}(G^F,r)'$ of $\mathrm{IBr}(G^F,r)$ by setting

$$\mathrm{IBr}(G^F,r)' = \{\varphi \in \mathrm{IBr}(G^F,r) : \varphi|_{U^F} \text{ contains a linear character of } U^F\}.$$

On the basis of looking at various examples, it seems that $\mathrm{IBr}(G^F,r)'$ contains 'most' of the Brauer characters of G^F but, unless G is of type A_l and F is the standard Frobenius map, $\mathrm{IBr}(G^F,r)'$ is strictly smaller than $\mathrm{IBr}(G,r)$. We note that an irreducible Brauer character of degree relatively prime to p is necessarily in $\mathrm{IBr}(G^F,r)'$.

THEOREM 5.3. *Assuming the notation introduced above, let $\varphi_1,\ldots,\varphi_s$ denote the different irreducible Brauer characters in $\mathrm{IBr}(G^F,r)'$ that occur as r-modular constituents of $1_{B^F}^{G^F}$, with corresponding non-zero multiplicities $m_1,\ldots,m_s$. Then, provided that $(U^*)^F$ is the derived group of U^F, we have*

$$m_1+\cdots+m_s \leq 2^t|W^F|,$$

where t is defined as in Lemma 5.1 (and t satisfies $t \leq l$).

Proof. Suppose that $\varphi_i|_{U^F}$ contains the linear character λ of U^F with non-zero multiplicity. Clifford's theorem implies that all the T^F-conjugates of λ occur as constituents of $\varphi_i|_{U^F}$ with the same multiplicity. Thus, we may assume that λ is one of the characters $\lambda_1, \ldots, \lambda_{2^t}$ that are taken as representatives of the 2^t orbits of T^F acting on the linear characters of U^F. For $1 \leq j \leq 2^t$, let n_{ij} denote the multiplicity of λ_j as a constituent of $\varphi_i|_{U^F}$. It follows that the multiplicity with which λ_j occurs as a constituent of π is

$$m_1n_{1j}+\cdots+m_sn_{sj}$$

and this number is at most $|W^F|$ by Lemma 5.2. If we now set

$$m_i' = \sum_{j=1}^{2^t} n_{ij}$$

for $1 \leq i \leq s$ and sum over the 2^t orbits of linear characters, we obtain

$$m_1m_1'+\cdots+m_sm_s' \leq 2^t|W^F|$$

and this certainly proves the desired result. ■

Theorem 5.3 implies that the decomposition number of any member of $\mathrm{IBr}(G^F, r)'$ with respect to an irreducible complex character in π is bounded by a function of the rank l of G and is independent of q (under our usual assumption concerning the derived group of U^F). While the restriction of our investigation to members of $\mathrm{IBr}(G^F, r)'$ is a disadvantage of our method, we note that the problem we encountered in Section 2 shows that considerations of this kind may be relevant. For, taking G^F to be $\mathrm{PU}(3, q)$, with $q \equiv 3 \bmod 4$ and $r = 2$, the only non-trivial irreducible constituent of π is the Steinberg character, whose decomposition number with respect to the Brauer character φ of degree $q(q-1)$ is not known presently. This Brauer character is not in $\mathrm{IBr}(G^F, 2)'$.

Corollary 5.4. *Let G be a simple algebraic group of adjoint type over an algebraically closed field of characteristic p, where p is an odd prime, and let $F : G \to G$ be a Frobenius map. Let K be an algebraically closed field of characteristic 2. Then, except possibly in the case that $p = 3$ and G^F is of type ${}^2E_6(q)$, the number of simple self-dual KG^F-modules of degree relatively prime to p that are not of quadratic type is bounded by a function of the rank l of G.*

We finish this section by showing that when G is of type A_l and F is the standard Frobenius map induced by $x \to x^q$, where q is a power of the prime p, we have $\mathrm{IBr}(G^F, r)' = \mathrm{IBr}(G^F, r)$ for any prime r different from p. In order to do this, it suffices to prove the following somewhat more general result.

Theorem 5.5. *Let q be a power of a prime p and let r be a prime different from p. Let P be a Sylow p-subgroup of $\mathrm{GL}(n, q)$ and let φ be an irreducible Brauer character of $\mathrm{GL}(n, q)$ modulo r. Then $\varphi|_P$ contains a linear Brauer character of P.*

Proof. We use a simple modification of an argument of Gelfand and Graev, [5], who proved a version of our proposed theorem for complex characters of $\mathrm{GL}(n, q)$. We set G_n equal to $\mathrm{GL}(n, q)$. Consider the subgroup Γ_n of G_n that fixes a given non-zero vector in the permutation action of G_n on the underlying n-dimensional vector space over $\mathbf{F}_q$. It is well known that Γ_n is the semi-direct product of an elementary abelian p-group T_n, say, of order q^{n-1}, by the general linear group G_{n-1} of degree $n-1$. Now as G_n acts transitively on the non-zero vectors, we have $|G_n : \Gamma_n| = q^n - 1$ and thus Γ_n contains a Sylow p-subgroup of G_n, which we may take to be P. To prove Theorem 5.5, it will clearly be enough to show that if φ is an element of $\mathrm{IBr}(\Gamma_n, r)$, $\varphi|_P$ contains a linear Brauer character of P. We proceed to prove this by induction on n. The case when $n = 1$ is trivial.

We may identify the irreducible Brauer characters modulo r of T_n with the complex characters of T_n. According to [16, Section 3], G_{n-1} has two orbits on the complex characters of T_n, one consisting of the trivial character, the other consisting of the $q^{n-1}-1$ non-trivial linear characters. The stabilizer in G_{n-1} of a non- trivial linear character is isomorphic to Γ_{n-1}, and we shall identify the stabilizer with Γ_{n-1}. Let φ be an irreducible Brauer character of Γ_n modulo r and let λ be an irreducible constituent of $\varphi|_{T_n}$. We consider two cases. First of all, suppose that λ is trivial. Then T_n is in the kernel of φ and thus φ is the inflation to Γ_n of an irreducible Brauer character φ', say, of the quotient group Γ_n/T_n, which is isomorphic to G_{n-1}. By induction, the restriction of φ' to a Sylow p-subgroup of G_{n-1} contains a linear character of the group and then it follows that the inflation of φ' to Γ_n contains a linear character of P when restricted to P. Thus we have proved our theorem in this case.

Secondly, suppose that λ is non-trivial. Then the stabilizer subgroup $J(\lambda)$, say, of λ in Γ_n is the semi-direct product $T_n\Gamma_{n-1}$. Clifford's theorem shows that there is an irreducible Brauer character, ψ, say, of $J(\lambda)$ lying over λ such that $\varphi = \psi^{\Gamma_n}$. Now as $J(\lambda)$ is a semi-direct product of T_n with Γ_{n-1}, we know from the discussion following Lemma 5.1 that we may extend λ to a linear character $\hat{\lambda}$ of $J(\lambda)$. It is then straightforward to prove that ψ must be a product $\hat{\lambda}\inf(\vartheta)$, where $\inf(\vartheta)$ denotes the inflation to $J(\lambda)$ of an irreducible Brauer character ϑ of Γ_{n-1}. By induction, the restriction of ϑ to a Sylow p-subgroup of Γ_{n-1} contains a linear character of the Sylow subgroup and it follows that $\inf(\vartheta)|_P$ contains a linear character of P. Thus ψ_P also contains a linear character of P. Finally, Mackey's theorem implies that the restriction of $\varphi = \psi^{\Gamma_n}$ to $J(\lambda)$ contains a summand equal to ψ and we deduce that our theorem also holds in this second case. ■

We mention a corollary of this result which, although not directly relevant to the subject matter of this paper, shows that the behaviour exhibited in the 2-modular decomposition of the Steinberg character of $\mathrm{PU}(3,q)$ for $q \equiv 3 \bmod 4$, when a decomposition number is at least 2, cannot occur for $\mathrm{GL}(n,q)$.

Corollary 5.6. *Let q be a power of a prime p and let St denote the Steinberg character of $\mathrm{GL}(n,q)$. Then for any prime r, the decomposition numbers of St modulo r are 0 or 1.*

Proof. The Steinberg character is well known to be irreducible modulo p and we may thus assume that $r \neq p$. Let P be a Sylow p-subgroup of $\mathrm{GL}(n,q)$. Since St vanishes on p-singular elements and has degree $|P|$, its restriction to P is the regular character of P. It follows that each linear character of P occurs with multiplicity 1 in $St|_P$. Suppose now that φ is an irreducible Brauer character occurring as an r-modular constituent of St with non-zero multiplicity d. By Theorem 5.5, $\varphi|_P$ contains some linear character λ, say. Thus λ occurs at least d times as a constituent of $St|_P$. It follows that $d = 1$. ■

Our final result is an application of the ideas of this section to the study of the theme of this paper.

Corollary 5.7. *Let q be a power of an odd prime and let K be an algebraically closed field of characteristic 2. Then the number of simple self-dual $K\mathrm{GL}(n,q)$-modules that are not of quadratic type is bounded by $2^n n!$ for all values of q.*

The bound $2^n n!$ above has been obtained by rather crude estimates and is only of qualitative interest. It is probable that a more realistic bound can be obtained by using Hecke algebra techniques.

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