PROBLEMS ON BRAUER CHARACTERS

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Abstract

In this paper we focus on problems of irreducible p-Brauer characters and state conjectures based on many examples which we computed. Many of the asked questions hold true for p-solvable groups, but their answers in general seem to require a much deeper understanding than we have at the moment. The questions are dealing with degrees, Hilbert divisors, divisibility, height-zero irreducible Brauer characters, number of irreducible Brauer characters in a p-block, and Cartan invariants. For instance, one of the main conjectures states that an irreducible Brauer character has Hilbert divisor 1 if and only if the character lies in a p-block of defect zero. This is true for p-solvable groups since in this case there is a relation between Hilbert divisors and vertices. However, even if a p-block of a non-p-solvable group contains only two irreducible Brauer characters we do not have an idea how to attack the problem. We hope that the conjectures and questions which we state in this paper inspire to further research.

Keywords: Degree, p-block, height-zero Brauer character, Hilbert divisor, Cartan invariant

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1. Introduction

Throughout the paper p is always a prime and G a finite group. By B we denote a p-block of G of defect d and by $C = C_B = (c_{\alpha\beta})$ its Cartan matrix. Let $\mathrm{IBr}_p(G)$ and $\mathrm{IBr}_p(B)$ be the set of irreducible p-Brauer characters of G, resp. of a p-block B of G, where we use

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a p-modular splitting system (K, R, k). Here R is a complete discrete valuation ring with unique maximal ideal πR, K is the quotient field of R of characteristic 0 and $k = R/\pi R$ the residue field of characteristic $p > 0$. We put $k(B) = |\operatorname{Irr}(B)|$ and $l(B) = |\operatorname{IBr}_p(B)|$. Let $n_{p'}$ denote the p'-part of $n \in \mathbb{N}$ and $G_{p'} = \{g \mid g \in G,\ g \text{ is a } p'\text{-element}\}$ the set of p-regular elements of G. We briefly write p^a for $|G|_p$. Furthermore, the height $\operatorname{ht}(\psi)$ of an irreducible ordinary or Brauer character ψ in B is defined by $\psi(1)_p = p^{a-d+\operatorname{ht}(\psi)}$, where d is the defect of B. Finally, for $\varphi \in \operatorname{IBr}_p(G)$, let $vx(\varphi)$ denote the vertex of the kG-module which affords φ, and Φ_φ denote the ordinary character of the projective cover of the module associated to φ. For an ordinary character χ we denote by χ° its restriction to the set of p'-elements.

2. Degrees

In [4, Problem 15] Richard Brauer stated the following problem: Does there exist a characterization of the dimension of the Jacobson radical by group-theoretical properties? Clearly,

$$|G| - \dim \mathrm{J}(kG) = \sum_{\varphi \in \operatorname{IBr}_p(G)} \varphi(1)^2,$$

where $\mathrm{J}(kG)$ denotes the Jacobson radical of the group algebra kG. Thus, Brauer's problem is concerned with the degrees of the irreducible Brauer characters. If G has a normal Sylow p-subgroup P, then the irreducible Brauer characters of G may be considered as the irreducible ordinary characters of the p'-group G/P. Thus we immediately get

$$|G|_{p'} = \sum_{\varphi \in \operatorname{IBr}_p(G)} \varphi(1)^2.$$

Conjecture 2.1. [30] *For all finite groups, we have*

$$|G|_{p'} \leq \sum_{\varphi \in \operatorname{IBr}_p(G)} \varphi(1)^2,$$

with equality if and only if G has a normal Sylow p-subgroup.

If G has a p-complement, then Conjecture 2.1 has an affirmative answer as already shown in [30], or in [21, Lemma 2.1]. Using a reduction to quasi-simple groups, which has been proved by Tong-Viet in [29], Malle presented in [13] a proof for all finite groups in the case $p = 2$. More on the conjecture can be found in [16].

3. Hilbert divisors

Definition 3.1. [10] A Brauer character β (not necessarily irreducible) is called *quasi-projective* if

$$\beta = \sum_{\varphi \in \mathrm{IBr}_p(G)} a_\varphi \Phi_\varphi^\circ$$

with $a_\varphi \in \mathbb{Z}$.

Definition 3.2. [10] Let B be a p-block of defect d and let $\varphi \in \mathrm{IBr}_p(B)$. We denote the smallest integer $n \in \mathbb{N}_0$ such that

$$p^n \varphi$$

is quasi-projective by $a(\varphi)$ and call $p^{a(\varphi)}$ the *Hilbert divisor* of φ.

As already shown in [10, Theorem 2.1], we always have $a(\varphi) \leq d$. Clearly, if φ belongs to a p-block of defect zero, then $a(\varphi) = 0$.

Conjecture 3.3. *If $a(\varphi) = 0$, then φ belongs to a p-block of defect zero.*

If $\varphi \in \mathrm{IBr}_p(B)$, where the block B is of defect d, then $a(\varphi) \geq d - \mathrm{ht}(\varphi)$. This is due to Dickson's Theorem [20, Corollary 2.14], showing that the p-part $|G|_p$ of the order of G divides the degree of any quasi-projective Brauer character. Thus, if $\mathrm{ht}(\varphi) \leq d$, then $a(\varphi) = 0$ implies $\mathrm{ht}(\varphi) = 0$, and according to [10, Proposition 2.10] φ belongs to a p-block of defect zero.

Note that for p-solvable groups we always have $a(\varphi) = d - \mathrm{ht}(\varphi)$ by [10, Proposition 2.7] and Fong's dimensional formula $\Phi_\varphi(1) = |G|_p \varphi(1)_{p'}$ [20, Corollary 10.14]. But for general groups the strong inequality $a(\varphi) > d - \mathrm{ht}(\varphi)$ may happen. For instance, if $G = J_1$, the first simple Janko group, then there exists an irreducible Brauer character φ of degree 56 in the principal 2-block for which

$$1 = a(\varphi) > d - \mathrm{ht}(\varphi) = 3 - 3 = 0.$$

So far, Conjecture 3.3 has been verified in may situations; for instance
a) if the group G is p-solvable or the defect group is cyclic [10].
b) if $p = 2$ and the defect group is abelian [31].

Question 3.4. *Does $a(\varphi) = d$ always implies that* $\mathrm{ht}(\varphi) = 0$?

For $\varphi \in \mathrm{IBr}_p(G)$, we denote by $c(\varphi)$ the complexity $c(S)$ of the kG-module S which affords φ. Recall that according to Alperin and Evans

[1], the complexity $c(S)$ is the smallest non-negative integer s such that such that

$$\lim_{n\to\infty} \frac{\dim P_n}{n^s} = 0$$

where $\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to S \to 0$ is a minimal projective resolution of S. According to [2, Theorem 5.1.1] S is a projective kG-module if and only if $c(S) = 0$.

Proposition 3.5. *If G is a p-solvable group and $\varphi \in \mathrm{IBr}_p(G)$, then $c(\varphi) \leq a(\varphi)$ with equality if and only if $vx(\varphi)$ is an elementary abelian p-group.*

Proof. By [3, Theorem 5.1], we have

$$c(\varphi) = r(vx(\varphi))$$

where $r(vx(\varphi))$ denotes the rank of $vx(\varphi)$. On the other hand, by [10, Proposition 2.8], we have

$$p^{a(\varphi)} = |vx(\varphi)|.$$

Thus we obtain

$$p^{a(\varphi)} = |vx(\varphi)| \geq p^{r(vx(\varphi))} = p^{c(\varphi)}$$

and the proof is complete. □

Note that Proposition 3.5 answers Conjecture 3.3 again for p-solvable groups.

Remark 3.6. There are examples with $a(\varphi) < c(\varphi)$. For instance, let $G = J_1$ be again the first simple Janko group and let $p = 2$. Then for $\varphi \in \mathrm{IBr}_2(G)$ with $\varphi(1) = 56$ one easily sees that $vx(\varphi)$ equals the Sylow 2-subgroup of G which is elementary abelian of order 8. Furthermore, $a(\varphi) = 1$. On the other hand we have $c(\varphi) = 2$, according to a private communication by John Carlson.

Next observe that

$$\beta = \sum_{\varphi \in \mathrm{IBr}_p(B)} b_\varphi \varphi = \sum_{\varphi \in \mathrm{IBr}_p(B)} a_\varphi \Phi_\varphi^\circ$$

with $b_\varphi \in \mathbb{N}_0$ and $a_\varphi \in \mathbb{Z}$ is equivalent to $Ca \geq 0$ where C is the Cartan matrix of B and $a = (a_\varphi)$, see [10, p. 508]. For $l = |\,\mathrm{IBr}_p(B)|$, let

$$\mathrm{cone}(C) = \{a \in \mathbb{Z}^l \mid Ca \geq 0\}$$

denote the rational cone of C. Its image under the map

$$\rho : \mathbb{Z}^l \ni a \mapsto \sum_{\varphi \in \mathrm{IBr}_p(B)} a_\varphi \Phi_\varphi^\circ$$

is exactly the set of quasi-projective Brauer characters. According to [27, Theorem 16.4], cone(C) has a unique minimal Hilbert basis B which is finite. Note that a Hilbert basis of cone(C) is a finite set $\{a_1, \ldots, a_h\} \subseteq$ cone(C) such that each vector in cone(C) is a non-negative integral combination of $\{a_1, \ldots, a_h\}$. According to the above it is natural to call $\rho(\mathsf{B})$ *the Hilbert basis of* B (with respect to C) and put $\mathrm{H}_p(B) = \rho(\mathsf{B})$. By [10, Theorem 2.1], we have

$$\{p^{a(\varphi)}\varphi \mid \varphi \in \mathrm{IBr}_p(B)\} \subseteq \mathrm{H}_p(B).$$

Conjecture 3.7. *If* $l = |\,\mathrm{IBr}_p(B)| \geq 2$*, then*

$$\{p^{a(\varphi)}\varphi \mid \varphi \in \mathrm{IBr}_p(B)\} \subset \mathrm{H}_p(B).$$

For p-solvable groups and blocks with cyclic defect groups Conjecture 3.7 holds true (see [10, Lemma 3.4]).

Conjecture 3.3 is obviously true if $l(B) = 1$. Even if $l(B) = 2$ we do not know the answer in general. The smallest example leads to the following question. Observe that a positive answer also contradicts Conjecture 3.7.

Question 3.8. Can the matrix

$$C = \begin{pmatrix} a & bp^d \\ bp^d & cp^d \end{pmatrix},$$

where $ac - b^2p^d = 1$ and $p \nmid a, b, c$, be the Cartan matrix of a p-block B of defect d?

If Question 3.8 has a positive answer, then $c > 1$ as the next example shows.

Example 3.9. Let B be a p-block of defect d with $l(B) = 2$. Let $\mathrm{IBr}_p(B) = \{\varphi, \psi\}$. By changing the notation of the irreducible Brauer characters if necessary, we may assume that $a(\psi) = d$. If $a(\varphi) = 0$, then $\mathrm{H}_p(B) = \{\varphi, p^d\psi\}$.
One easily sees that the Cartan matrix is given by

$$C = \begin{pmatrix} a & bp^d \\ bp^d & cp^d \end{pmatrix}$$

where $ac - b^2p^d = 1$ and p does not divide a, b, c. Suppose that $c = 1$, hence

$$C = \begin{pmatrix} b^2p^d + 1 & bp^d \\ bp^d & p^d \end{pmatrix}.$$

It follows

$$(*) \qquad \sum_{\chi} d_{\chi\varphi}^2 = c_{\varphi\varphi} = b^2p^d{+}1, \sum_{\chi} d_{\chi\psi}^2 = c_{\psi\psi} = p^d, \sum_{\chi} d_{\chi\varphi}d_{\chi\psi} = c_{\varphi\psi} = bp^d.$$

Thus

$$\sum_{\chi}(d_{\chi\varphi} - bd_{\chi\psi})^2 = c_{\varphi\varphi} - 2bc_{\varphi\psi} + c_{\psi\psi} = (b^2p^d + 1) - 2b^2p^d + b^2p^d = 1.$$

This shows that there exists χ_0 such that $|d_{\chi_0\varphi} - bd_{\chi_0\psi}| = 1$ and $d_{\chi\varphi} = bd_{\chi\psi}$ for $\chi \neq \chi_0$. Now

$$d_{\chi_0\varphi}^2 = (bd_{\chi_0,\psi} \pm 1)^2 = b^2d_{\chi_0,\psi}^2 \pm 2bd_{\chi_0\psi} + 1$$

together with the conditions in (*) forces $d_{\chi_0\psi} = 0$. Thus

$$|G|_p \mid \varphi(1) \mid \chi_0(1)$$

and χ_0 is of defect 0, by [20, Theorem 3.18], a contradiction.

4. Divisibility of the Brauer characters Φ_φ°

Definition 4.1. If $\varphi \in \mathrm{IBr}_p(B)$, then n_φ denotes the largest $n \in \mathbb{N}$ for which there exists a Brauer character β (not necessarily irreducible) such that

$$n\beta = \Phi_\varphi^\circ.$$

Note that n_φ is the greatest common divisor of the Cartan invariants of the row associated to Φ_φ. Thus $n_\varphi \mid \det C$, hence n_φ is a power of p, by Brauer [5, Chap. IV, Theorem 3.9]. We may call n_φ the divisor of Φ_φ, and write $n_\varphi = p^{b(\varphi)}$.

In [10] it has been shown that there exists an integer matrix A such that

$$CA = \begin{pmatrix} p^{a(\varphi_1)} & 0 & \dots & 0 \\ 0 & p^{a(\varphi_2)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & p^{a(\varphi_l)} \end{pmatrix}$$

where $p^{a(\varphi_i)}$ denotes the Hilbert divisor of $\varphi_i \in \mathrm{IBr}_p(B)$. Thus we get $b(\varphi) \leq a(\varphi)$. Hence $b(\varphi) \leq d$, by [10]. Clearly, if $|\mathrm{IBr}_p(B)| = 1$, then $b(\varphi) = d$ for the only irreducible Brauer character in B.

Conjecture 4.2. *If $l(B) = |\mathrm{IBr}_p(B)| \geq 2$, then*

$$b(\varphi) < d$$

for all $\varphi \in \mathrm{IBr}_p(B)$.

In the case that all Cartan invariants of B are bounded by p^d, Conjecture 4.2 has an affirmative answer, as the next result shows. In particular, the conjecture holds true for p-solvable groups by [20, Theorem 10.22].

Theorem 4.3. *Let B be a p-block of defect d with $l(B) \geq 2$ and let $\varphi \in \mathrm{IBr}_p(B)$. If all Cartan invariants of B are bounded by p^d, i.e., $c_{\alpha\beta} \leq p^d$ for all $\alpha, \beta \in \mathrm{IBr}_p(B)$, then*

$$c_{\varphi\alpha} < p^d$$

for at least one $\alpha \in \mathrm{IBr}_p(B)$. In particular, $b(\varphi) < d$.

Proof. Suppose that $c_{\varphi\alpha} = p^d$ for all $\alpha \in \mathrm{IBr}_p(B)$. By the Cauchy-Schwarz inequality, we get

$$p^{2d} = c_{\varphi\alpha}^2 \leq c_{\varphi\varphi} c_{\alpha\alpha} \leq p^d p^d = p^{2d}.$$

Thus we have equality everywhere, and $c_{\alpha\alpha} = p^d$ for all α. It follows that

$$C = \begin{pmatrix} p^d & p^d & \dots & p^d \\ p^d & p^d & * & * \\ \vdots & * & \ddots & * \\ p^d & * & * & p^d \end{pmatrix},$$

where the first row corresponds to φ. Obviously, the determinant of the second principal minor

$$C = \begin{pmatrix} p^d & p^d \\ p^d & p^d \end{pmatrix}$$

is 0, a contradiction to the fact that C is positive definite. □

Finally, observe that $b(\varphi) = d - 1$ may happen as the principal 2-block of the alternating group A_5 shows.

5. Number of irreducible Brauer characters in a block

For Brauer characters α, β, the scalar product $\langle \cdot\,, \cdot \rangle^\circ$ is defined by

$$\langle \alpha, \beta \rangle^\circ = \frac{1}{|G|} \sum_{x \in G_{p'}} \alpha(x)\beta(x^{-1}).$$

If C is the Cartan matrix of the p-block B, then

$$C^{-1} = (\langle \alpha, \beta \rangle^\circ) \qquad\qquad (\alpha, \beta \in \mathrm{IBr}_p(B))$$

according to [20, Theorem 2.13]. Clearly, the entries in C^{-1} are rational numbers since C is an integral matrix. Furthermore $\langle \alpha, \beta \rangle^\circ \in \mathbb{Z}$ for all

$\beta \in \mathrm{IBr}_p(B)$ if and only if the Hilbert divisor of α is equal to 1, i.e., $a(\varphi) = 0$ (see [10, Proposition 2.12]).

Furthermore, observe that

$$|G|_p \langle 1_G, 1_G \rangle^\circ = \frac{|G_{p'}|}{|G|_{p'}}.$$

Conjecture 5.1. (Murai [18]) If $l(B_0) = |\,\mathrm{IBr}_p(B_0)|$, where B_0 denotes the principal p-block of G, then

$$l(B_0) \leq |G|_p \langle 1_G, 1_G \rangle^\circ.$$

Conjecture 5.2. [12] *If B is any p-block of defect d, then*

$$l(B) \leq p^d \langle \varphi, \varphi \rangle^\circ$$

for all $\varphi \in \mathrm{IBr}_p(B)$.

Even in the case of p-solvable groups, it remains unknown whether Conjecture 5.1 and Conjecture 5.2 are true.

In his paper 'Problems on characters: solvable groups' [23] Navarro stated the following.

Problem 5.3. [23, Problem 3.4] Let B be a p-block of defect d. If $\chi \in \mathrm{Irr}(B)$, is it true that

$$l(B) \leq p^d \langle \chi, \chi \rangle^\circ ?$$

The author also pointed out that, by a proof of Sambale [23, Theorem 3.5], a positive answer to Problem 5.3 would imply Brauer's $k(B)$-conjecture, i.e.,

$$k(B) = |\,\mathrm{Irr}(B)| \leq p^d.$$

Recall that the sectional p-rank $s(D)$ of a finite p-group D is the maximal rank of an elementary abelian group isomorphic to L/K for some subgroups $K \lhd L$ of D. For instance, the Sylow 2-subgroups of A_5 and S_5 are isomorphic to $C_2 \times C_2$ and D_8, respectively, whose sectional p-ranks are both 2. In [15] Malle and Robinson conjectured the following.

Conjecture 5.4. [15, Conjecture 1] *For any finite group we have $l(B) \leq p^{s(B)}$ where $s(B)$ denotes the sectional p-rank of the defect group of B.*

Note that the authors proved $l(B) < p^{s(B)}$ for p-solvable groups if the defect of B is not zero.

Let G be the first simple Janko group J_1, and B_0 the principal 2-block of G. The GAP library [28] shows that there are $\alpha_0, \beta_0 \in \mathrm{IBr}_p(B_0)$ such that $\langle \alpha, \beta \rangle^\circ = 2$. By the next result, we see that if G is p-solvable, then

$$|\langle \alpha, \beta \rangle^\circ| \leq 1$$

for all $\alpha, \beta \in \mathrm{IBr}_p(B)$.

Lemma 5.5. *If $\alpha, \beta \in \mathrm{IBr}_p(G)$ are both liftable, then*

$$|\langle \alpha, \beta \rangle^\circ| \leq 1.$$

Furthermore, $|\langle \alpha, \beta \rangle^\circ| < 1$ if $\alpha \neq \beta$.

Proof. By the Chauchy-Schwarz inequality, we have

$$|\langle \alpha, \beta \rangle^\circ|^2 \leq \langle \alpha, \alpha \rangle^\circ \langle \beta, \beta \rangle^\circ.$$

Let $\chi \in \mathrm{Irr}(G)$ be a lift of α, i.e., $\chi^\circ = \alpha$. Then

$$1 = \langle \chi, \chi \rangle = \langle \alpha, \alpha \rangle^\circ + \frac{1}{|G|} \sum_{x\ p\text{-singular}} \chi(x)\overline{\chi(x)}.$$

Since both parts on the right hand side are real and non-negative, we obtain $0 \leq \langle \alpha, \alpha \rangle^\circ \leq 1$. As the same holds true for β, we get $|\langle \alpha, \beta \rangle^\circ| \leq 1$. Now suppose that for $\alpha \neq \beta$ we have $|\langle \alpha, \beta \rangle^\circ| = 1$. Thus, by equality in the Cauchy-Schwarz inequality we obtain $\alpha = z\beta$ for some complex number z, a contradiction, since α and β are linearly independent. □

Question 5.6. Does there always exist $\alpha \in \mathrm{IBr}_p(B)$ such that

$$|\langle \alpha, \beta \rangle^\circ| \leq 1$$

for all $\beta \in \mathrm{IBr}_p(B)$. In particular, $\sum_{\beta \in \mathrm{IBr}_p(B)} |\langle \alpha, \beta \rangle^\circ| \leq l(B)$.

Finally we may ask whether there is a nice upper bound for $|\langle \alpha, \beta \rangle^\circ|$ where $\alpha, \beta \in \mathrm{IBr}_p(B)$.

6. Irreducible height zero Brauer characters in a block

Recently, the proof of Brauer's height zero conjecture (already formulated by Brauer in 1955) has been completed in [14]. It says that all irreducible ordinary characters of a p-block B have height zero if and only if the defect group of B is abelian.

For irreducible height zero Brauer characters it is hard to say what an analog might be. For instance, the principal 3-block of M_{11} has an abelian Sylow 3-subgroup, but also an irreducible Brauer character whose degree is divisible by 3. The extended alternating group $3.\mathrm{Alt}_7$

has a non-abelian Sylow 3-subgroup, but all irreducible Brauer characters in the principal 3-block are of height zero.

Proposition 6.1. (Rothschild [26]) *If the defect group of the block B is cyclic, then all irreducible Brauer characters in B are of height zero.*

Proposition 6.2. *Let G be a finite p-solvable group. Then all irreducible Brauer characters in the principal block B_0 of G are of height zero if and only if $G = \mathrm{O}_{p',p,p'}(G)$.*

Proof. Note that $\mathrm{O}_{p'}(G)$ is the kernel of the principal block, by [7, Chap VII, Theorem 14.8]. Furthermore, the defect group of B_0 is a Sylow p-subgroup, say P, of G.

Suppose that all irreducible Brauer characters of B_0 are of height zero. Write $\overline{G} = G/\mathrm{O}_{p'}(G)$. Since G is p-solvable, it follows that $C_{\overline{G}}(\mathrm{O}_p(\overline{G})) \leq \mathrm{O}_p(\overline{G})$. By [5, Chap. V, Corollary 3.11], $\overline{G}$ has only one block, i.e., the principal block B_0. Hence all irreducible Brauer characters of $\overline{G}$ are of height zero, i.e., of p'-degree. By Michler's Theorem [17, Theorem 2.4], $\overline{G}$ has a normal Sylow p-subgroup.

Conversely, suppose that $G = \mathrm{O}_{p',p,p'}(G)$. Since $P\,\mathrm{O}_{p'}(G)/\mathrm{O}_{p'}(G)$ is a normal Sylow p-subgroup of $G/\mathrm{O}_{p'}(G)$, the irreducible Brauer characters in the principal p-block B_0 of G may be seen as ordinary characters of the p'-group $G/\mathrm{O}_{p',p}(G)$. Thus they are of p'-degree, hence of height zero. □

Olsson's conjecture [25] asserts that the number of irreducible ordinary characters of height zero in a p-block of a finite group with defect group D is at most $|D : D'|$. For the corresponding number of Brauer characters of height zero, we put forward the following.

Conjecture 6.3. *Let G be a finite group and let B be p-block of G with defect group D. If $l_0(B)$ denotes the number of the set of irreducible Brauer characters of height zero of B, then $l_0(B) \leq |D : \Phi(D)|$, where $\Phi(D)$ denotes the Frattini subgroup of D.*

Note that the Malle-Robinson conjecture (Conjecture 5.4) implies Conjecture 6.3 in the case that the defect group is abelian since

$$l_0(B) \leq l(B) \leq p^{s(B)} = |D : \Phi(D)|.$$

Furthermore, by the same arguments as in [15, Proposition 3.1] we see that $l_0(B) < |D : \Phi(D)|$ if D is dihedral, (generalised) quaternion or semi-dihedral.

In our next result, we show that the Malle-Robinson conjecture also implies Conjecture 6.3 when the defect group is normal.

Proposition 6.4. *Let B be a p-block of a finite group G with a normal defect group D. If the Malle-Robinson conjecture holds true, then* Conjecture 6.3 *holds true.*

Proof. The proof follows the lines of [8, Theorem 1]. Let $H = DC_G(D) \trianglelefteq G$, b a p-block of H covered by B, T the inertia group of b in G, and B_1 the Clifford correspondent of B in T. By [19, Chap. V, Theorem 5.10], D is a defect group of B_1, and there is a bijection of height-zero Brauer characters in B and B_1 (see also an analog of [20, Theorem 9.14 (d)] for Brauer characters). In particular, $l_0(B) = l_0(B_1)$.

Denote $\overline{G} = G/\Phi(D)$. Since the irreducible Brauer characters of G have kernels containing D, we may view them as the irreducible Brauer characters of $\overline{G}$. Let $l_0(B_1)$ be the set of irreducible height-zero Brauer characters of B_1. We claim that, for any $\varphi, \psi \in l_0(B_1)$, when viewing them as Brauer characters of $\overline{T}$, their corresponding characters $\overline{\varphi}$ and $\overline{\psi}$ belong to the same p-block of $\overline{T}$. Note that B is regular by [19, Chap. V, Lemma 5.14], and so D is a Sylow p-subgroup of T by [19, Chap. V, Theorem 5.16]. Hence $\overline{D}$ is a Sylow p-subgroup of $\overline{T}$. In particular, φ and ψ are characters of p'-degree. For the sum $\hat{g}$ of the conjugacy class C_g of a p'-element $g \in T$ we have

$$\omega_\varphi(\hat{g}) := \frac{|T : C_T(g)|\varphi(g)}{\varphi(1)} \equiv \frac{|T : C_T(g)|\psi(g)}{\psi(1)} \bmod \pi,$$

since φ and ψ belong to the same block B_1. This follows from the fact that φ and ψ are liftable since T is p-solvable. Note that

$$|C_g| = \frac{|T : C_T(g)|}{|\bar{T} : C_{\bar{T}}(\bar{g})|}|C_{\bar{g}}|.$$

Clearly $\omega_\varphi(\hat{g}) = 0$, if g is not of maximal defect. Thus we may assume that g is of maximal defect. As a consequence $p \nmid \frac{|T:C_T(g)|}{|\bar{T}:C_{\bar{T}}(\bar{g})|} \in \mathbb{N}$ and we deduce

$$\omega_{\bar{\varphi}}(\hat{\bar{g}}) \equiv \omega_{\bar{\psi}}(\hat{\bar{g}}) \bmod \pi$$

for all p'-elements in T. This says that $\overline{\varphi}$ and $\overline{\psi}$ belong to the same p-block, as claimed (see page 48 of [20]). Thus $l_0(B_1) = l_0(\overline{B_1})$, where $\overline{B_1}$ is the p-block of $\overline{T}$ dominated by B_1. If the Malle-Robinson conjecture is true then since $\overline{D}$ is elementary abelian, it follows that $l_0(\overline{B_1}) \leq l(\overline{B_1}) \leq |\overline{D}|$. Thus $l_0(B_1) \leq \overline{D}$, finishing the proof. □

Theorem 6.5. *Let G be a finite p-solvable group and B a p-block of G. Then* Conjecture 6.3 *holds true.*

Proof. Let D be a defect group of B, $N = N_G(D)$ and b the Brauer correspondent of B in N. According to [5, Ch. X, Theorem 1.8], we see that the irreducible Brauer characters of B of height zero are exactly those of B with vertex D. By [24, Theorem 4.1], we have $l_0(B) = l(b)$. Furthermore D is contained in the vertex of any irreducible Brauer character of b (i.e. contained in the associated module), by [5, Chap. III, Corollary 4.13]. On the other hand, such a vertex must be contained in D, by [19, Chap. V, Theorem 1.9]. Thus $l(b) = l_0(b)$. Now the statement of the Theorem follows from Proposition 6.4 and the trueness of the Malle-Robinson conjecture for p-solvable groups [15, Theorem 2]. □

7. Cartan invariants

In [5, Chap. IV, Question (VII)] Walter Feit asked:

Is there a function of p and d which bounds $c_{\alpha\beta}$ for $\alpha, \beta \in \mathrm{IBr}_p(B)$ in a p-block of defect d?

By Landrock's counterexample [9], we know that in general we do not have

$$c_{\alpha,\beta} \leq p^d$$

for $\alpha, \beta \in \mathrm{IBr}_p(B)$. In [22] Navarro and Sambale pointed out that they are not aware of any counterexample to

$$c_{\alpha,\beta} \leq p^{2d}$$

for $\alpha, \beta \in \mathrm{IBr}_p(B)$.

A crucial example might be the following.

Example 7.1. Let B_0 denote again the principal 2-block of the first simple Janko group J_1. Then

$$c_{\alpha,\beta} \leq p^{a(\alpha)+a(\beta)}$$

for $\alpha, \beta \in \mathrm{IBr}_p(B_0)$ with equality if $\alpha \neq 1_G \neq \beta$.

Question 7.2. *Do we always have*

$$c_{\alpha,\beta} \leq p^{a(\alpha)+a(\beta)}$$

for $\alpha, \beta \in \mathrm{IBr}_p(B)$?

Remark 7.3. Let G be p-solvable and let $\alpha, \beta \in \mathrm{IBr}_p(B)$. Note that $p^{a(\alpha)} = |vx(\alpha)|$, by [10, Proposition 2.7]. By [6] we have

$$c_{\alpha\beta} \leq \max\{|vx(\alpha)|, |vx(\beta)|\} = p^{\max\{a(\alpha),a(\beta)\}} \leq p^{a(\alpha)+a(\beta)}.$$

Now suppose that we have equality everywhere and the block B has defect $d > 0$. Since Conjecture 3.3 has an affirmative answer for

p-solvable groups, we may deduce a contradiction since either $a(\alpha)$ or $a(\beta)$ must be zero. Thus in the case of p-solvable groups and blocks of positive defect Question 7.2 has an affirmative answer with a strong inequality.

Example 7.1 shows that we can not do better as we asked in Question 7.2. Furthermore, if the question has an affirmative answer, then we have

$$c_{\alpha,\beta} \leq p^{2d},$$

by [10, Theorem 2.1].

Finally, for $\alpha \in \mathrm{IBr}_p(G)$, we define $\widetilde{\alpha}$ by

$$\widetilde{\alpha}(x) = \begin{cases} p^{a(\alpha)}\alpha(x), & \text{if } x \text{ is } p\text{-regular}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\widetilde{\alpha}$ is a generalized ordinary character, by ([10], Corollary 2.3), and

$$\langle \widetilde{\alpha}, \widetilde{\beta} \rangle = p^{a(\alpha)+a(\beta)} \langle \alpha, \beta \rangle^\circ.$$

Thus, we may reformulate Question 7.2 in the following way.

Question 7.4. *Do we always have*

$$c_{\alpha\beta} \leq \frac{\langle \widetilde{\alpha}, \widetilde{\beta} \rangle}{\langle \alpha, \beta \rangle^\circ}$$

if $\langle \alpha, \beta \rangle^\circ \neq 0$*?*

Lemma 7.5. *If* $\langle \alpha, \beta \rangle^\circ \neq 0$*, then* $|\langle \widetilde{\alpha}, \widetilde{\beta} \rangle| \geq p^{\max\{a(\alpha),a(\beta)\}}$

Proof. For $\alpha \in \mathrm{IBr}_p(B)$ we have

$$p^{a(\alpha)}\alpha = \sum_{\beta \in \mathrm{IBr}_p(B)} a_\beta \Phi_\beta$$

where $a_\beta \in \mathbb{Z}$, by definition of Hilbert divisors. Thus,

$$p^{a(\alpha)} \langle \alpha, \beta \rangle^\circ = a_\beta,$$

which implies

$$|\langle \alpha, \beta \rangle^\circ| = \frac{|a_\beta|}{p^{a(\alpha)}} \geq \frac{1}{p^{a(\alpha)}}.$$

By symmetry, we are done. ☐

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