On higher Frobenius-Schur indicators

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Abstract

Similarly to the Frobenius-Schur indicator of irreducible characters we consider higher Frobenius-Schur indicators $\nu_{p^n}(\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^{p^n})$ for primes p and $n \in \mathbb{N}$, where G is a finite group and χ is a generalized character of G. It turns out that these invariants give answers to interesting questions in representation theory. In particular, we give several characterizations of groups via higher Frobenius-Schur indicators.

Keywords. irreducible character, higher Frobenius-Schur indicator, Brauer character, permutation module

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1 Introduction

Throughout the paper let G be a finite group, p a prime and k a splitting field of characteristic p for G and all its subgroups. In some of the proofs we take k as the residue field of a p-modular system $(K, \Re, k = \Re/\wp)$ where $p \in \wp = \pi\Re$. By a we denote the p-exponent of G, i.e., $a = \min\{n \mid n \in \mathbb{N},\ g^{p^n} = 1 \text{ for all } p\text{-elements } g \in G\}$. Let $G_{p^n} = \{x \in G \mid x^{p^n} = 1\}$ for $n \in \mathbb{N}$ and $G_{p'} = \{g \in G \mid p \nmid \mathrm{ord}(g)\}$. Then the k-vector space kG_{p^n} (k not necessarily of characteristic p) becomes a kG-module via the conjugation action by G. If $\{x_1, x_2, \cdots, x_r\}$ is a set of representatives of the G-conjugacy classes of G_{p^n}, we obtain the natural decomposition

$$kG_{p^n} \cong (k_{C_G(x_1)})^G \oplus (k_{C_G(x_2)})^G \oplus \cdots \oplus (k_{C_G(x_r)})^G.$$

In the case of even characteristic, i.e., $p = 2$, the so-called involution module kG_2 has been studied to some extend by several authors. By ([5], Corollary 4.6), we have

$$1 + |G_2| = \sum_{\chi} \nu_2(\chi)\chi(1)$$

where the sum runs through the set of irreducible complex characters of G and $\nu_2(\chi)$ denotes the Frobenius-Schur indicator of χ. Robinson started in [18] the control of the projective summands of kG_2 by properties of the Frobenius-Schur indicator. In a series of papers [11, 12, 13, 14] Murray continued the investigations of the structure of kG_2. The block decomposition of kG_2 has been given in [9] by Martínez-Pérez and the second author, using a natural splitting of the cohomology module $H^1(G, \Lambda^2(kG))$, which turned out to be isomorphic to the involution module.

In this note we continue the investigations of the structure of kG_{p^n}, in particular for odd primes p. Here the higher Frobenius-Schur indicators are coming in.

2 The module kG_{p^n} and higher Frobenius-Schur indicators

Let $\mathrm{Irr}(G)$ and $\mathrm{IBr}_p(G)$ denote the set of irreducible complex, resp. irreducible Brauer characters of G in characteristic p. By $1 \in \mathrm{Irr}(G)$, resp. $1 \in \mathrm{IBr}_p(G)$ we always mean the trivial character, resp. the trivial Brauer character. Finally, by $[\cdot,\cdot]$ we denote the usual scalar product on the ring of generalized characters.

For $\ell \in \mathbb{N}$, we put

$$\vartheta_\ell(g) = |\{h \in G | h^\ell = g\}|.$$

ϑ_ℓ is a class function, and clearly $\vartheta_\ell = \sum_{\chi\in\mathrm{Irr}(G)} \nu_\ell(\chi)\chi$, where $\nu_\ell(\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^\ell) \in \mathbb{Z}$ (see [5], Chapter 4). We call $\nu_\ell(\chi)$ the *higher Frobenius-Schur indicator* of χ for $\ell \neq 2$. By ([5], Theorem 4.5), we have $\nu_2(\chi) = -1, 0,$ or 1 for $\chi \in \mathrm{Irr}(G)$. But for $l \neq 2$, the situation turns out to be more subtle and in fact, there is even no absolute bound for $\nu_\ell(\chi)$ (see [5], Problems 4.9).

For a generalized character ψ we extend the definition of higher Frobenius-Schur indicators by putting

$$\nu_\ell(\psi) = \frac{1}{|G|}\sum_{g\in G}\psi(g^\ell) = [\psi^{(\ell)}, 1]$$

where $\psi^{(\ell)} = \psi(g^\ell)$ for all $g \in G$. Finally, for a Brauer character $\varphi \in \mathrm{IBr}_p(G)$, we always denote by Φ_φ the projective indecomposable character of G associated to φ. Furthermore, 1_G denotes the trivial character in characteristic p and 0 as well, and we shortly write Φ_1 for Φ_{1_G}. Finally, for a character χ, we put $\chi^\circ = \chi|_{G_{p'}}$.

In ([11], Lemma 3), Murray proved that $\varphi \in \mathrm{IBr}_2(G)$ occurs in the Brauer character of the involution module over an algebraically closed field of characteristic 2 with multiplicity $\nu_2(\Phi_\varphi)$. His proof works not only for G_2, but also for all G_{p^n}.

Theorem 2.1 *If Λ_{p^n} is the complex character of $\mathbb{C}G_{p^n}$ with $n \in \mathbb{N}$, then*

$$\Lambda^\circ_{p^n} = \sum_{\chi\in\mathrm{Irr}(G)} \nu_{p^n}(\chi)\chi^\circ = \sum_{\varphi\in\mathrm{IBr}_p(G)} \nu_{p^n}(\Phi_\varphi)\varphi$$

where

$$\nu_{p^n}(\Phi_\varphi) \leq \nu_{p^a}(\Phi_\varphi) \leq \Phi_\varphi(1)\frac{|G|-|G_{p'}|}{|G|} + \delta_{\varphi 1_G}.$$

In particular, $\nu_{p^n}(\Phi_\varphi)$ is the multiplicity of φ in the p-Brauer character of kG_{p^n}, hence $\nu_{p^n}(\Phi_\varphi) \in \mathbb{N}_0$ and $\nu_{p^n}(\Phi_1) > 0$.

Proof: By definition of kG_{p^n}, we have $\Lambda_{p^n}(g) = |G_{p^n} \cap C_G(g)|$ for all $g \in G$. Note that the map $\varsigma : G_{p'} \to G_{p'}$ defined by $\varsigma(g) = g^{p^n}$ is a bijection for any $n \in N$. If $g \in G_{p'}$, then for any $h \in G$ such that $h^{p^n} = g$, the p'-part $h_{p'} = g^{\frac{1}{p^n}}$ is uniquely determined by g and $(h_p)^{p^n} = 1$. Hence $h_{p^n} \in G_{p^n} \cap C_G(g)$. On the other hand, for any $x \in G_{p^n} \cap C_G(g)$, we have $(x(g_{p'})^{\frac{1}{p^n}})^{p^n} = g_{p'}$. So it follows that $|G_{p^n} \cap C_G(g)| = |\{h \in G \mid h^{p^n} = g\}|$ if $g \in G_{p'}$. From $|\{h \in G \mid h^{p^n} = g\}| = \sum_{\chi\in\mathrm{Irr}(G))} \nu_{p^n}(\chi)\chi$ we conclude that

$$\begin{aligned}
\Lambda^\circ_{p^n} &= \textstyle\sum_{\chi\in\mathrm{Irr}(G))} \nu_{p^n}(\chi)\chi^\circ \\
&= \textstyle\sum_{\chi\in\mathrm{Irr}(G))} \sum_{\varphi\in\mathrm{IBr}_p(G)} \nu_{p^n}(\chi) d_{\chi\varphi}\varphi \\
&= \textstyle\sum_{\varphi\in\mathrm{IBr}_p(G)} \nu_{p^n}(\Phi_\varphi)\varphi.
\end{aligned}$$

Finally, we have to prove the upper bound for $\nu_{p^n}(\phi_\varphi)$. Since $\nu_{p^n}(\phi_\varphi)$ is the multiplicity of φ in $\Lambda^\circ_{p^n}$, we obviously have $\nu_{p^n}(\phi_\varphi) \leq \nu_{p^a}(\phi_\varphi)$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
\nu_{p^a}(\Phi_\varphi) &= \tfrac{1}{|G|}\textstyle\sum_{g\in G} \Phi_\varphi(g^{p^a}) \\
&= \tfrac{1}{|G|}\textstyle\sum_{g\in G_{p'}} \Phi_\varphi(g^{p^a}) + \tfrac{1}{|G|}\sum_{g\in G\setminus G_{p'}} \Phi_\varphi(g^{p^a}) \\
&= \delta_{\varphi 1_G} + \tfrac{1}{|G|}\textstyle\sum_{g\in G\setminus G_{p'}} \Phi_\varphi(g^{p^a}) \\
&= \delta_{\varphi 1_G} + \tfrac{1}{|G|}\textstyle\sum_{g\in G\setminus G_{p'}} \Phi_\varphi(g^{p^a}) \\
&\leq \delta_{\varphi 1_G} + \tfrac{|G|-|G_{p'}|}{|G|}\Phi_\varphi(1)
\end{aligned}$$

Finally, $\nu_{p^n}(\Phi_1) \neq 0$ since kG_{p^n} is a permutation module. □

The following immediate consequence of Theorem 2.1 generalizes ([5], Corollary 4.6).

Corollary 2.2 *If t is the number of elements of order p in G, then*

$$1+t = \sum_{\chi\in\mathrm{Irr}(G)} \nu_p(\chi)\chi(1).$$

Proof: By Theorem 2.1, we have $1+t = |G_p| = \sum_{\chi\in\mathrm{Irr}(G)} \nu_p(\chi)\chi(1)$. $\square$

Note that for $n \in \mathbb{N}$ and $g \in G_{p'}$, we have $\Lambda_{p^n}(g) = |C_G(g)_{p^n}|$. Thus the class function defined by $g \mapsto |C_G(g)_{p^n}|$ is a Brauer character, by Theorem 2.1. In general, δ_{p^n} is not a character, but a generalized character. Furthermore,

$$|C_G(g)_{p^n}| = \sum_{\varphi\in\mathrm{IBr}_p(G)} \nu_{p^n}(\Phi_\varphi)\varphi(g)$$

for $g \in G_{p'}$.

In a first step to understand the structure of kG_{p^n}, we need to know more about the higher Frobenius-Schur indicators.

Proposition 2.3 *For a projective indecomposable character Φ of G and all $n \in \mathbb{N}$, we have $\nu_{p^n}(\Phi) \equiv [\Phi, 1]$ mod p. In particular, $\nu_{p^n}(\Phi) \equiv \nu_p(\Phi)$ mod p.*

Proof: By ([5], Problems 4.7), we know that $\Phi^{p^n} - \Phi^{(p^n)} = p^n\mu$ for some character μ, hence $[\Phi^{p^n}, 1] \equiv [\Phi^{(p^n)}, 1]$ mod p. Since $\nu_{p^n}(\Phi) = [\Phi^{(p^n)}, 1]$, it suffices to show that $[\Phi^{p^n}, 1] \equiv [\Phi, 1]$ mod p. Using the p-modular system $(K, \Re, k = \Re/\wp)$, we get

$$\begin{aligned} [\Phi^{p^n}, 1] = \tfrac{1}{|G|}\textstyle\sum_{g\in G} \Phi^{p^n}(g) &= \tfrac{1}{|G|}\textstyle\sum_{g\in G_{p'}} \Phi^{p^n}(g) \\ &\equiv \tfrac{1}{|G|}\textstyle\sum_{g\in G_{p'}} \Phi(g^{p^n}) \quad \bmod \wp \\ &= \tfrac{1}{|G|}\textstyle\sum_{g\in G_{p'}} \Phi(g) \quad \bmod \wp \\ &= [\Phi, 1] \quad \bmod \wp. \end{aligned}$$

Thus $[\Phi^{p^n}, 1] \equiv [\Phi, 1]$ mod $\wp \cap \mathbb{Z} = p\mathbb{Z}$; i.e., $[\Phi^{p^n}, 1] \equiv [\Phi, 1]$ mod p. $\square$

Theorem 2.4 *If G is a group of even order, then $2 \mid \nu_{2^n}(\Phi_1)$ for all $n \in \mathbb{N}$.*

Proof: By Theorem 2.1, we have $\sum_{\varphi\in\mathrm{IBr}_2(G)} \nu_{2^n}(\Phi_\varphi)\varphi(1) = |G_{2^n}|$.

If $S = G \setminus G_{2^n}$, then for all $x \in S$, we have $x \neq x^{-1} \in S$. Thus $2 \mid |S|$. Since $|G| = |G_{2^n}| + |S|$, it follows that $2 \mid |G_{2^n}|$. Furthermore, again by Theorem 2.1, $\nu_{2^n}(\Phi_\varphi)$ is a non-negative integer, and from $\overline{\Phi_\varphi} = \Phi_{\overline{\varphi}}$ we deduce that $\nu_{2^n}(\Phi_{\bar{\varphi}}) = \nu_{2^n}(\Phi_\varphi)$.

According to Fong's lemma ([15], Theorem 2.31), we have $2 \mid \varphi(1)$ for $1 \neq \varphi = \overline{\varphi} \in \mathrm{IBr}_2(G)$. Thus

$$\begin{aligned} |G_{2^n}| &= \textstyle\sum_{\varphi \in \mathrm{IBr}_2(G)} \nu_{2^n}(\Phi_\varphi)\varphi(1) \\ &= \nu_{2^n}(\Phi_1) + \textstyle\sum_{1 \neq \bar{\varphi} = \varphi \in \mathrm{IBr}_2(G)} \nu_{2^n}(\Phi_\varphi)\varphi(1) + \textstyle\sum_{\bar{\varphi} \neq \varphi \in \mathrm{IBr}_2(G)} \nu_{2^n}(\Phi_\varphi)\varphi(1) \ , \\ &\equiv \nu_{2^n}(\Phi_1) \quad \mod 2 \end{aligned}$$

and the assertion follows since $2 \mid |G_{2^n}|$. □

Note that Theorem 2.4 does not hold true for Φ_φ with $\varphi \neq 1_G$, and in the case p odd, also not for Φ_1.

Example 2.5 a) Let $\varphi = \overline{\varphi} \in \mathrm{IBr}_2(G)$ be a real valued Brauer character of 2-defect 0 where $2 \mid |G|$. Then $\Phi_\varphi = \overline{\Phi_\varphi} \in \mathrm{Irr}(G)$. Hence $|\nu_2(\Phi_\varphi)| = 1$. Acually, $\nu_2(\Phi_\varphi) = 1$, by ([3], Proposition 1.1).
b) Let $p = 3$ and $G = \mathrm{S}_3$. One easily computes $\Phi_1 = 1_G + \chi$ with $\chi(1) = 2$. It follows $\nu_3(\Phi_1) = \frac{1}{6}\sum_{g \in G} \Phi_1(g^3) = \frac{1}{6}\sum_{g \in G}(1 + \chi(g^3)) = \frac{1}{6}(3+3+3+1+1+1) = 2$. □

Theorem 2.4 has an interesting consequence for the decomposition matrix $D = (d_{\chi\varphi})$ (see Corollary 2.7). In order to state it we need the following result which can be deduced from a paper of Quillen [16], and Thompson [19] as well. For the readers convenience, we present a proof which already appeared in the unpublished thesis [21] of the second author. In order to state and prove it we use again a p-modular system $(K, \Re, k = \Re/\wp)$ where $p \in \wp = \pi\Re$. Furthermore, let $\bar{\ }: \Re \longrightarrow \Re/\wp = k$ be the natural epimorphism.

Lemma 2.6 *Let V be a KG-module with a non-degenerate G-invariant symplectic bilinear form $b(\cdot,\cdot)$. Then V has an $\Re G$-lattice M, and there exists a chain of kG-modules $0 \subseteq R \subset \overline{M}$ such that $\overline{M}/R$ and R (latter if not 0) carry non-degenerate G-invariant symplectic bilinear forms.*

Proof: Let N be an $\Re G$-lattice of V, in particular $N \otimes_\Re K = V$. Multiplying the bilinear form by a suitable scalar we may assume that

$$b(N, N) \subseteq \Re, \quad \text{but} \quad b(N, N) \not\subseteq \pi\Re.$$

We put

$$\widehat{N} = \{v \mid v \in V,\, b(v, N) \subseteq \Re\}.$$

Let $n_1, \dots, n_t$ be free generators of the free $\Re$-module N. Since $V = NK$ and $V \cong V^*$, there are $v_1, \dots, v_t \in V$ such that $b(v_i, n_j) = \delta_{ij}$. Thus $v_1, \dots, v_t$ are free generators of $\widehat{N}$ as an $\Re$-module which shows that $\widehat{N}$ is an $\Re G$-lattice of V. Let M be a maximal element in the set

$$\{N' \mid N \subseteq N' \subseteq \widehat{N},\ N' \text{ an } \Re G\text{-module},\ b(N', N') \subseteq \Re\}.$$

Note that M exists since $\widehat{N}$ is a noetherian $\Re G$-module. We define a G-invariant symplectic k-bilinear form $c(\cdot, \cdot)$ on $\overline{M} = M/\pi M$ by

$$c(\overline{m}, \overline{m'}) = \overline{b(m, m')}$$

for $m, m' \in M$. Clearly, c is well defined. Since $b(M, M) \not\subseteq \pi\Re$, the radical $R = \mathrm{rad}_c(\overline{M})$ of c is a proper kG-submodule of $\overline{M}$. Thus $\overline{M}/R$ carries a non-degenerate G-invariant symplectic bilinear form.

In the case $R \neq 0$ we show that also R carries a non-degenerate G-invariant symplectic bilinear form. Note that $U = \{m \mid m \in M, b(m, M) \subseteq \pi\Re\}$ is the preimage of R in M. Suppose that $b(U, U) \subseteq \pi^2\Re$. Hence $b(\pi^{-1}U, \pi^{-1}U) \subseteq \Re$. Since $N \subseteq M \subseteq \pi^{-1}U$, we have $b(\pi^{-1}U, M) \subseteq \Re$. It follows $M \subseteq \pi^{-1}U \subseteq \widehat{N}$ and by maximality of M, we obtain $M = \pi^{-1}U$, hence $U = \pi M$ which means that $R = 0$ and we are done.

Thus $b(U, U) \subseteq \pi\Re$, but $b(U, U) \not\subseteq \pi^2\Re$. On $R = U/\pi M$ we define a G-invariant symplectic bilinear form $d(\cdot, \cdot)$ by

$$d(u + \pi M, u' + \pi M) = \overline{\pi^{-1}b(u, u')}$$

for $u, u' \in U$. The bilinear form d is well defined since

$$d(\pi M, U) = \overline{\pi^{-1}b(\pi M, U)} = \overline{b(M, U)} = 0.$$

To finish the proof it remains to show that d is non-degenerate. The preimage of the radical $\mathrm{rad}_d(U)$ of d in R is

$$U_0 = \{u \mid u \in U, b(u, U) \subseteq \pi^2\Re\}.$$

Clearly, $\pi M \subseteq U_0$, hence $M \subseteq \pi^{-1}U_0$. Now $b(\pi^{-1}U_0, \pi^{-1}U_0) \subseteq \Re$ implies $\pi^{-1}U_0 \subseteq \widehat{N}$. The maximality of M forces $\pi^{-1}U_0 = M$. Thus $\mathrm{rad}_d R = 0$, and the proof is complete. □

Corollary 2.7 *If $p = 2$ and $2 \mid |G|$, then*

(i) $2 \mid d_{\chi 1_G}$ *for* $\chi = \overline{\chi} \in \mathrm{Irr}(G)$ *with* $\nu_2(\chi) = -1$.

(ii) $2 \mid \sum d_{\chi 1_G}$ *where the sum runs over all* $\chi = \overline{\chi} \in \mathrm{Irr}(G)$ *with* $\nu_2(\chi) = 1$.

Proof: (i) The condition $\nu_2(\chi) = -1$ says that the module V affording χ carries a non-degenerate G-invariant symplectic form. By Lemma 2.6, V has a lattice M such that its reduction $\overline{M} = M/\wp M$ has a submodule R where R (if not 0) and $\overline{M}/R$ have a non-degenerate G-invariant symplectic bilinear form. Since the trivial module does not allow a non-zero G-invariant symplectic bilinear form, its multiplicity in $\overline{M}$ must be even according to the argument at the end of chapter 2 in [23].
(ii)

$$\Phi_1 = \sum_{\substack{\chi=\overline{\chi} \\ \nu_2(\chi)=1}} d_{\chi 1_G}\chi + \sum_{\substack{\chi=\overline{\chi} \\ \nu_2(\chi)=-1}} d_{\chi 1_G}\chi + \sum_{\substack{\chi\neq\overline{\chi} \\ \nu_2(\chi)=0}} d_{\chi 1_G}(\chi + \overline{\chi}).$$

Applying (i) we get

$$\nu_2(\Phi_1) \equiv \sum_{\substack{\chi=\overline{\chi} \\ \nu_2(\chi)=1}} d_{\chi 1_G} \mod 2.$$

Since $2 \mid \nu_2(\Phi_1)$, by Theorem 2.4, the assertion follows. □

Example 2.8 For $G = \mathrm{SL}(2,5)$, we have the following numbers in agreement with Corollary 2.7:

$d_{\chi 1_G}$	1	1	1	0	1	0	0	0	2
$\nu_2(\chi)$	1	1	1	1	1	-1	-1	-1	-1

□

Recall that $\nu_2(\chi) \in \{0, 1\}$ for $\chi \in \mathrm{Irr}(G)$ of 2-defect zero. Actually $\nu_2(\chi) = 0$ if χ is not real-valued and otherwise $\nu_2(\chi) = 1$ (see Example 2.5).

Proposition 2.9 *Let p be any prime dividing $|G|$ and $n \in \mathbb{N}$. If $\chi \in \mathrm{Irr}(G)$ is of p-defect zero, then* $0 \leq \nu_{p^n}(\chi) \leq \chi(1) - \frac{|G_{p'}|}{|G|_{p'}}\chi(1)_{p'} \leq \chi(1) - \chi(1)_{p'}$.

Proof: By Theorem 2.1, we have

$$0 \leq \nu_{p^n}(\chi) \leq \nu_{p^a}(\chi) \leq \chi(1)\frac{|G| - |G_{p'}|}{|G|},$$

since $\chi = \Phi_\varphi$ for $\varphi \neq 1_G$. Furthermore, $|G|_p = \chi(1)_p$, by ([15], Theorem 3.18), and $|G|_{p'} \mid |G_{p'}|$, by ([2], Proposition 15.9). Thus

$$\chi(1)\frac{|G| - |G_{p'}|}{|G|} = \chi(1) - \chi(1)_{p'}\frac{|G_{p'}|}{|G|_{p'}} \leq \chi(1) - \chi(1)_{p'}.$$

□

Example 2.10 Let $G = C_2 \times \mathrm{A}_4$. One easily checks that for $p = 3$, the group G has two irreducible characters χ and ψ, both of degree 3, of 3-defect zero and self-dual. For one of the characters, say χ, we have $\nu_3(\chi) = 2$, for the other one $\nu_3(\psi) = 0$. This shows in particular that the lower and upper bound in Proposition 2.9 are sharp, and $\nu_{p^n}(\chi) = 0$ for p odd does not imply $\chi \neq \overline{\chi}$. □

For the reader's convenience we recall the following well-known result (see for instance [17] or [7]).

Lemma 2.11 *If each $g \in G_{p'}$ is centralized by all p-elements of G, then*

$$G = \mathrm{O}_{p'}(G) \times \mathrm{O}_p(G).$$

Proof: For a prime $q \neq p$, let Q be a Sylow q-subgroup of G and let P be a Sylow p-subgroup. Clearly,

$$N = \langle Q^g \mid g \in G \text{ and } q \neq p \rangle$$

is a normal subgroup of G and N satisfies the assumption of the lemma. If $N < G$, then, by induction, $N = \mathrm{O}_{p'}(N) \times \mathrm{O}_p(N)$. Since G/N is a p-group and centralizes $\mathrm{O}_{p'}(N)$ we are done. Thus we may assume that $N = G$. Now P is central in G, hence a normal subgroup. By the Schur-Zassenhaus Theorem, P has a complement U in G, which is centralized by P, and the proof is complete. □

For the next result, recall that p^a is the largest p-power dividing $|G|$.

Theorem 2.12 *The following conditions are equivalent.*

(i) $\nu_{p^a}(\Phi_\varphi) = 0$ *for all* $1_G \neq \varphi \in \mathrm{IBr}_p(G)$.

(ii) $G = \mathrm{O}_{p'}(G) \times \mathrm{O}_p(G)$.

Proof: (i) $\Longrightarrow$ (ii) Recall that

$$|C_G(g)_{p^a}| = \sum_{\varphi \in \mathrm{IBr}_p(G)} \nu_{p^a}(\Phi_\varphi)\varphi(g),$$

for all $g \in G_{p'}$. Thus, by the assumption in (i), we get

$$|G_{p^a}| = \nu_{p^a}(\Phi_1) = |C_G(g)_{p^a}|,$$

for all $g \in G_{p'}$. Hence each p'-element of G is centralized by all p-elements of G and the assertion follows by Lemma 2.11.

(ii) $\Longrightarrow$ (i) For $1_G \neq \varphi \in \mathrm{IBr}_p(G)$ we have

$$\begin{aligned} \nu_{p^a}(\Phi_\varphi) &= \tfrac{1}{|G|}\textstyle\sum_{g\in G}\Phi_\varphi(g^{p^a}) = |\operatorname{O}_p(G)|\tfrac{1}{|G|}\sum_{g\in \operatorname{O}_{p'}(G)}\Phi_\varphi(g^{p^a}) \\ &= |\operatorname{O}_p(G)|\tfrac{1}{|G|}\textstyle\sum_{g\in \operatorname{O}_{p'}(G)}\Phi_\varphi(g) = |\operatorname{O}_p(G)|\,[\Phi_\varphi, 1_G] = 0. \end{aligned}$$

□

Proposition 2.13 *Suppose that $p \mid |G|$. If $\nu_p(\Phi_\varphi) = 0$ for all $1_G \neq \varphi \in \mathrm{IBr}_p(G)$, then G has a non-trivial central p-subgroup. In particular, $\operatorname{O}_p(G) \neq 1$.*

Proof: As in the previous proof we have $|C_G(g)_p| = \sum_{\varphi\in \mathrm{IBr}_p(G)} \nu_p(\Phi_\varphi)\varphi(g)$, for all $g \in G_{p'}$. Thus, by the assumption,

$$(*) \qquad\qquad |C_G(g)_p| = \nu_p(\Phi_1),$$

for all $g \in G_{p'}$, and in particular $|C_G(g)_p| = |G_p|$ (taking $g = 1$). Note that $|\nu_p(\Phi_1)| \neq 1$, since $p \mid |G|$. If P is a Sylow p-subgroup of G, then $P_0 = \Omega_1(Z(P)) \neq 1$. Thus P_0 centralizes P, and by $(*)$, all Sylow q-subgroups of G for $q \neq p$. This shows that $P_0 \leq Z(G)$ and the proof is complete. □

3 Groups whose irreducible characters all have non-zero Frobenius-Schur indicators

Suppose that G satisfies the following two conditions:

(i) All irreducible complex characters are real-valued, hence $\nu_2(\chi) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$.

(ii) For all nonlinear $\chi \in \mathrm{Irr}(G)$, we have $\nu_2(\chi) = -1$.

In [22], the second author proved that these conditions force G to be a 2-group. For instance, the quaternion group Q_8 and elementary abelian 2-groups satisfy both conditions. In the following we prove an analogous result for higher Frobemius -Schur indicators, by using the classification of finite simple groups. In order to prove Theorem 3.4 we need the following observations.

Lemma 3.1 *Let G be a q-group with $q \neq p$. Then $\nu_p(\chi) = 0$ for all $1_G \neq \chi \in \mathrm{Irr}(G)$.*

Proof: We have $\nu_p(\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^p) = \frac{1}{|G|}\sum_{g\in G}\chi(g) = [\chi, 1_G] = \delta_{\chi 1_G}$. □

Lemma 3.2 *If G is an abelian p-group, but not elementary abelian, then there exists a linear chracter χ with $\nu_p(\chi) = 0$.*

Proof: Since G is not elementary abelian, there exists a normal subgroup N such that $G = \langle gN \rangle$ is cyclic of order p^2. Let $\chi \in \mathrm{Irr}(G)$ with $\mathrm{Ker}\,\chi = N$ and $\chi(gN) = \epsilon$ where ϵ is a primitive complex p^2-th root of unity. If $\omega = \epsilon^p$, then

$$\nu_p(\chi) = \frac{|N|}{|G|}\sum_{i=0}^{p^2}\chi(g^{pi}N) = \frac{1}{p^2}\cdot p(1+\omega+\cdots+\omega^{p-1}) = 0.$$

□

Lemma 3.3 *If G has a p-block B which is not of maximal defect, then G has a non-linear irreducible character χ with $\nu_p(\chi) \geq 0$.*

Proof: Clearly, all $\chi \in \mathrm{Irr}(B)$ are non-linear since the defect of B is not maximal. By Theorem 2.1, we have $\nu_p(\Phi_\varphi) \geq 0$ for all $\varphi \in \mathrm{IBr}_p(G)$. Writing $\Phi_\varphi = \sum_{\chi\in\mathrm{Irr}(B)} d_{\chi\varphi}\chi$, we obtain

$$0 \leq \nu_p(\Phi_\varphi) = \sum_{\chi\in\mathrm{Irr}(B)} d_{\chi\varphi}\nu_p(\chi).$$

Thus there exists $\chi \in \mathrm{Irr}(B)$ with $\nu_p(\chi) \geq 0$.

□

Theorem 3.4 *Let G be a finite group and p be an odd prime. Then G satisfies*

(i) $\nu_p(\chi) \neq 0$ *for all* $\chi \in \mathrm{Irr}(G)$ *and*

(ii) $\nu_p(\chi) < 0$ *for all nonlinear* $\chi \in \mathrm{Irr}(G)$

if and only if G *is an elementary abelian* p*-group.*

Proof: Clearly, if G is an elementary abelian p-group, then all irrecucible characters χ are linear and satisfy $\nu_p(\chi) = 1$. Thus G satisfies both conditions. Conversely, let G be a group satisfying (i) and (ii). Clearly, any factor group of G also satisfies the conditions. In order to prove that G is an elementary abelian p-group we use induction on the order of G. Let N be a minimal normal subgroup of G. Then, by induction, we get that G/N is an elementary abelian p-group. Furthermore N is the unique minimal normal subgroup, since otherwise G is (up to an isomorphism) a subgroup of $G/N_1 \times G/N_2$ which is an elementary abelian p-group, by induction. Now we may write

$$N = S_1 \times \cdots \times S_t \quad \text{with} \quad S = S_1 \cong S_i \quad \text{for all}\, i.$$

Case $p \nmid |S|$:
Thus $G = \mathrm{O}_{p'}\, P$ where P is an elementary abelian p-group. Suppose that there exists $x \in \mathrm{O}_{p'}$ with $|C_G(x)|_p < |P|$. Applying ([20], Theorem 1), we get that G has a p-block of non-maximal defect, contradicting Lemma 3.3. Thus $p \nmid |x^G|$ for all p'-elements x and we obtain

$$G = \mathrm{O}_{p'}(G) \times P,$$

by [17]. According to Lemma 3.1 we get $\mathrm{O}_{p'}(G) = 1$ and we are done.
Case $p \mid |N|$:
Thus $p \mid |S|$. We first suppose that the simple group S is non-abelian. Since all blocks of G have maximal defect by Lemma 3.3, all p-blocks of S have maximal defect. We prove that this is not true. If S is a group of Lie-type, then S has p-blocks of defect zero, by ([10], Theorem 5.1). If $S \cong A_n$, then S has again a p-block of defect zero in the case $p \geq 5$, by ([4], Corollary 1). In the case that $p = 3$ there exists a p-block of defect $d \leq \frac{a-1}{2}$, by ([1], Theorem 2), except $S = A_7$. But A_7 has a 3-block of defect 1. If S is sporadic, then S has a p-block of defect zero, by ([4], Corollary 2) unless $p = 3$ and $S \cong Suz$ or $S = Co_3$. In the two exceptional cases there exists a 3-block of defect 1. Thus S must be cyclic of order p and G is an extension of an elementary abelian p-group by an elementary abelian p-group. Clearly $N = G'$, since N is a minimal normal

subgroup and G is not abelian, by Lemma 3.2. Furthermore, since the action of the p-group G/N on the p-group N is irreducible, we get $|N| = p$. Since N is the unique minimal normal subgroup of G, we see that $Z(G) = N$ or $N < Z(G)$ and $|Z(G)| = p^2$. Thus we have to consider the following two cases:

(a) $G' = \Phi(G) = Z(G)$, i.e., G is extraspecial,

(b) $G' = \Phi(G) < Z(G)$ and $|Z(G)| = p^2$. (Such groups exist.)

First we consider the case (a): If $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$, then there exists $\mu \in \mathrm{Irr}(A)$ such that $\chi = \mu^G$ where A is a maximal normal subgroup of G and $\lambda = \mu|_N \neq 1_N$ (see [6], Kap. V, Satz 16.14)).

It follows that

$$\nu_p(\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^p) = \frac{1}{|G|}\sum_{g\in G}\chi(1)\lambda(g^p) = \frac{\chi(1)}{|G|}\left(|G_p| + \sum_{g\in G,\mathrm{ord}(g)=p^2}\lambda(g^p)\right).$$

By assumption, we have $\nu_p(\chi) < 0$, which forces that $\sum_{\substack{g\in G\\ \mathrm{ord}(g)=p^2}}\lambda(g^p)$ is a real negative number and

$$|\sum_{\substack{g\in G\\ \mathrm{ord}(g)=p^2}}\lambda(g^p)| > |G_p|.$$

Since G' is cyclic and $p > 2$, G is a regular p-group, by ([6], Kap. III, Satz 10.2). In particular, $\Omega_1(G) = \{g \in G \mid g^p = 1\}$, by ([6], Kap. III, Hauptsatz 10.8). It follows

$$\begin{aligned} |\Omega_1(G)| = |G_p| &< |\textstyle\sum_{\substack{g\in G\\ \mathrm{ord}(g)=p^2}}\lambda(g^p)| \leq \sum_{g\in G\setminus\Omega_1(G)}|\lambda(g^p)| \leq |G/\Omega_1(G)| \\ &= |\Omega_2(G)/\Omega_1(G)| \leq |\Omega_1(G)/\Omega_0(G)| = |\Omega_1(G)| - 1 \end{aligned}$$

where the last inequality comes from ([6], Kap. III, Satz 10.7). Thus we have a contradiction.

Dealing with the case (b), we choose $\lambda \in \mathrm{Irr}(Z(G))$ with $\lambda|_N \neq 1_N$. By ([6], Kap. V, Satz 6.3), the character λ has an extension μ to a maximal normal abelian subgroup A of G. Note that not all irreducible constituents of μ^G can be linear since otherwise N is in the kernel of μ^G. Thus, there exists a non-linear irreducible constituent χ of μ^G and we may argue as in (a) for χ to get the final contradiction. □

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